Section 1.5

Transformation of Astrometric Data and Associated Error Propagation

1.5. Transformation of Astrometric Data and Associated Error Propagation

1.5.1. Introduction

This section describes the following, generally applicable, transformations of the astrometric data which may be of interest to users of the Hipparcos and Tycho Catalogues (and as incorporated in the *Celestia 2000* software):

- from equatorial to ecliptic coordinates;
- from equatorial to galactic coordinates;
- from the catalogue epoch $T_0 = \text{J1991.25(TT)}$ to an arbitrary epoch T.

Although these or similar transformations are discussed in many textbooks, the corresponding transformation of covariances is usually not covered, and the epoch transformation not always treated at a level of approximation adequate for the present data.

Additionally, the calculation of rectangular space coordinates and space velocities in the equatorial and galactic systems is considered in Section 1.5.6. Section 1.5.7 describes some features of the relation between the reference system ICRS as materialised by the Hipparcos Catalogue, and the conventional coordinate system at J2000.0 previously realised by the FK5 Catalogue.

In this section the astrometric parameters are represented by vectors or column matrices $\mathbf{a}$, with subscripts indicating which coordinate system or epoch they refer to; the corresponding covariance matrices are denoted $\mathbf{C}$, cf. Equations 1.2.4 and 1.2.5. Specifically, $\mathbf{a}_0$ and $\mathbf{C}_0$ refer to the equatorial system at the common catalogue epoch $T_0 = \text{J1991.25(TT)}$, and thus correspond exactly to the data as given in the Hipparcos and Tycho Catalogues (Fields H/T8–9 and H/T11–28) as explained in Section 1.2.

In the ecliptic coordinates the astrometric parameters are denoted $(\lambda, \beta, \pi, \mu_{\lambda *}, \mu_\beta)$, or in abbreviated vector form $\mathbf{a}_\mathrm{K}$; similarly in the galactic system $(l, b, \pi, \mu_{l*}, \mu_b)$, or $\mathbf{a}_\mathrm{G}$ in vector form. Here, $\mu_{\lambda *} = \mu_\lambda \cos\beta$ and $\mu_{l*} = \mu_l \cos b$. The subscript $_0$ may be added if the data specifically refer to the catalogue epoch T_0.

1.5.2. General Error Propagation

In Section 1.2.8 a standard model of stellar motion was introduced, and other, more complex models are discussed in Section 2.3. The numerical fitting of such a model to observational data generally results in an estimate both of the parameter vector **a** and of its covariance matrix **C**. With n free parameters, these are expressed respectively as matrices of dimension $n \times 1$ and $n \times n$.

From a statistical viewpoint, **a** and **C** jointly provide a complete specification of the fitted model and its uncertainties only in the context of linear estimation and normal (Gaussian) error distributions. The estimation problems encountered in the Hipparcos and Tycho data analyses are seldom strictly linear, and sometimes strongly non-linear, and the error distributions are in practice never Gaussian. Perhaps the most dramatic consequence of non-linearity is the possible existence of grid-step errors, which in unfortunate cases may result in positional errors of the order of an arcsec while the formal standard errors remain in the milli-arcsec range. Additional statistics provided in the catalogue, such as the rejection rate and goodness-of-fit (Fields H29–30), provide some indication of the sometimes abnormal behaviour of data, but in practice no complete characterisation of the fit is possible. Nevertheless, in the vast majority of cases **a** and **C** provide an extremely useful approximation to complex reality.

If the parameters **a** are transformed into an alternative representation $\hat{\mathbf{a}} = \mathbf{f}(\mathbf{a})$ of the same dimension, then small errors in the parameters are transformed according to:

$$\Delta\hat{a}_i = \sum_j \frac{\partial f_i}{\partial a_j}\Delta a_j \qquad [1.5.1]$$

In matrix form this can be written:

$$\Delta\hat{\mathbf{a}} = \mathbf{J_f}(\mathbf{a})\Delta\mathbf{a} \qquad [1.5.2]$$

where $\mathbf{J_f}(\mathbf{a})$ is the Jacobian matrix of the transformation:

$$[\mathbf{J_f}]_{ij} = \frac{\partial f_i}{\partial a_j} \qquad [1.5.3]$$

evaluated at the point **a**. Now let $\Delta\mathbf{a}$ be the difference between the estimated and true parameter vectors. If the estimate is unbiased, then $\mathrm{E}(\Delta\mathbf{a}) = \mathbf{0}$, where E is the expectation operator, and the covariance of **a** is given by $\mathbf{C} = \mathrm{E}(\Delta\mathbf{a}\Delta\mathbf{a}')$, with the prime denoting matrix transposition. From Equation 1.5.2 it follows that $\hat{\mathbf{a}}$ is also unbiased, to the first order in the errors, and that its covariance is given by:

$$\hat{\mathbf{C}} = \mathbf{J_f}\,\mathbf{C}\,\mathbf{J}'_\mathbf{f} \qquad [1.5.4]$$

Equation 1.5.4 is the basis for the error propagation discussed below.

If the inverse function $\mathbf{f}^{-1}$ exists, then it is possible to transform the data set $[\,\hat{\mathbf{a}}\ \hat{\mathbf{C}}\,]$ back to the original form $[\,\mathbf{a}\ \mathbf{C}\,]$, and the two representations can be regarded as equivalent from the point of view of information content. A necessary condition for this is that $|\mathbf{J_f}| \neq 0$, in which case $\mathbf{J}_{\mathbf{f}^{-1}} = \mathbf{J}_\mathbf{f}^{-1}$. The transformations discussed here satisfy this condition.

1.5.3. Coordinate Transformations

In Section 1.2 the vectors representing celestial directions were effectively identified with the column matrices containing their components in the equatorial system; see for instance Equations 1.2.11, 1.2.15 and 1.2.19. When dealing with transformations between different coordinate systems it is useful to maintain the distinction between a vector as a physical entity and its numerical representation in some coordinate system. This requires that the basis vectors of the coordinate systems are explicitly introduced. The basis vectors in the equatorial system are here denoted $[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,]$, with $\mathbf{x}$ being the unit vector towards $(\alpha,\delta)=(0,0)$, $\mathbf{y}$ the unit vector towards $(\alpha,\delta)=(+90^\circ,0)$, and $\mathbf{z}$ the unit vector towards $\delta=+90^\circ$. The basis vectors in the ecliptic and galactic systems are respectively denoted $[\,\mathbf{x}_\mathrm{K}\ \mathbf{y}_\mathrm{K}\ \mathbf{z}_\mathrm{K}\,]$ and $[\,\mathbf{x}_\mathrm{G}\ \mathbf{y}_\mathrm{G}\ \mathbf{z}_\mathrm{G}\,]$. Thus, the arbitrary direction $\mathbf{u}$ may be written in terms of the equatorial, ecliptic and galactic coordinates as:

$$\mathbf{u}=[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,]\begin{pmatrix}\cos\delta\cos\alpha\\ \cos\delta\sin\alpha\\ \sin\delta\end{pmatrix}=[\,\mathbf{x}_\mathrm{K}\ \mathbf{y}_\mathrm{K}\ \mathbf{z}_\mathrm{K}\,]\begin{pmatrix}\cos\beta\cos\lambda\\ \cos\beta\sin\lambda\\ \sin\beta\end{pmatrix}=[\,\mathbf{x}_\mathrm{G}\ \mathbf{y}_\mathrm{G}\ \mathbf{z}_\mathrm{G}\,]\begin{pmatrix}\cos b\cos l\\ \cos b\sin l\\ \sin b\end{pmatrix}\qquad[1.5.5]$$

The transformation between the equatorial and ecliptic systems is given by:

$$[\,\mathbf{x}_\mathrm{K}\ \mathbf{y}_\mathrm{K}\ \mathbf{z}_\mathrm{K}\,]=[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,]\,\mathbf{A}_\mathrm{K}\qquad[1.5.6]$$

where:

$$\mathbf{A}_\mathrm{K}=\begin{pmatrix}1&0&0\\0&\cos\epsilon&-\sin\epsilon\\0&\sin\epsilon&\cos\epsilon\end{pmatrix}=\begin{pmatrix}1&0&0\\0&+0.917\,482\,062\,1&-0.397\,777\,155\,9\\0&+0.397\,777\,155\,9&+0.917\,482\,062\,1\end{pmatrix}\qquad[1.5.7]$$

$\epsilon=23^\circ\,26'\,21.448''=23.^\circ439\,291\,111\,1\ldots$ is the conventional value of the obliquity of the ecliptic (Table 1.2.2).

The transformation between the equatorial and galactic systems is given by:

$$[\,\mathbf{x}_\mathrm{G}\ \mathbf{y}_\mathrm{G}\ \mathbf{z}_\mathrm{G}\,]=[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,]\,\mathbf{A}_\mathrm{G}\qquad[1.5.8]$$

where the matrix $\mathbf{A}_\mathrm{G}$ relates to the definition of the galactic pole and centre in the ICRS system. Currently no definition of this relation has been sanctioned by the IAU. In order to provide an unambiguous transformation for users of the Hipparcos and Tycho Catalogues, the north galactic pole is here defined by the following celestial coordinates in the ICRS system:

$$\begin{aligned}\alpha_\mathrm{G}&=192.^\circ859\,48\\ \delta_\mathrm{G}&=+27.^\circ128\,25\end{aligned}\qquad[1.5.9]$$

The origin of galactic longitude is defined by the galactic longitude of the ascending node of the galactic plane on the equator of ICRS, which is here taken to be:

$$l_\Omega=32.^\circ931\,92\qquad[1.5.10]$$

Equations 1.5.9 and 1.5.10 are consistent with the previous (1960) definition of galactic coordinates to a level set by the quality of optical reference frames prior to Hipparcos.

The angles α_G, δ_G and l_Ω are to be regarded as exact quantities. From them, the transformation matrix A_G may be computed to any desired accuracy. To 10 decimal places the result is:

$$\mathbf{A}_G = \begin{pmatrix} -0.054\,875\,560\,4 & +0.494\,109\,427\,9 & -0.867\,666\,149\,0 \\ -0.873\,437\,090\,2 & -0.444\,829\,630\,0 & -0.198\,076\,373\,4 \\ -0.483\,835\,015\,5 & +0.746\,982\,244\,5 & +0.455\,983\,776\,2 \end{pmatrix} \qquad [1.5.11]$$

The current system of galactic longitudes and latitudes, $(l^{\mathrm{II}}, b^{\mathrm{II}})$, have been defined by adopting the coordinates of directions to the north galactic pole and the galactic centre, based on physical features in the Galaxy, with respect to the B1950 coordinate reference frame (A. Blaauw, C.S. Gum, J.L. Pawsey & G. Westerhout, 1960, *Mon. Not. R. Astron. Soc.*, 121, 123). It is clearly desirable that $(l^{\mathrm{II}}, b^{\mathrm{II}})$ computed from coordinates referred to the ICRS should be equal to those computed from coordinates referred to B1950. In principle this could be achieved by transforming the directions of the galactic axes first to the J2000 systems using standard precepts, and then to the ICRS, as realised by the Hipparcos Catalogue.

The main problem arises because the transformation from B1950 to J2000 consists not only of a pure rotation of axes due to changes in precession and equinox motion, but also a change in the convention by which stellar aberration is computed in the two systems. Prior to the adoption of the J2000 system, stellar aberration was based only on the circular component of the Earth's orbital velocity. The component depending on the eccentricity of the orbit, which can amount to 0.34 arcsec, and is approximately constant for a given direction over long intervals of time, was considered to be implicitly included in the mean catalogue positions of stars. Therefore if the galactic axes continue to be defined by the same physical features as in the 1960 definitions, they will no longer be mutually orthogonal. This is clearly unacceptable. Accordingly Murray (1989, *Astronomy & Astrophysics*, 218, 325) proposed that the axes defined in 1960 should be considered as absolute directions, unaffected by aberration; the angles given in Equations 1.5.9 and 1.5.10 were derived from his Equation 33 by rounding to five decimals of the degree.

The J2000 reference system is realised in practice by the FK5 Catalogue. The mean relation between this and the Hipparcos Catalogue involves a (time-dependent) rotation, which amounts to some 20 mas at the epoch J1991.25 (see Section 1.5.7). A precise and unambiguous determination of the difference in orientation between the two catalogues is however prevented by the much larger regional differences (~ 100 mas), and by colour and magnitude equations in FK5, which affect its orientation at the 20–30 mas level. It is therefore inappropriate to demand continuity with the previous definition at a level substantially below 100 mas. With this in mind, the orientation difference between ICRS and FK5 may be ignored in the present context.

The definition of the angles α_G, δ_G and l_Ω in Equations 1.5.9 and 1.5.10, using five decimals of the degree, implies a change in the directions of the principal axes, compared to the definition proposed by Murray (1989), by at most 18 mas. The ambiguity of the FK5/ICRS transformation does not warrant a higher numerical precision of the defining angles.

The ecliptic longitude and latitude are thus computed from:

$$\begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} = \mathbf{A}'_K \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \qquad [1.5.12]$$

and the galactic longitude and latitude from:

$$\begin{pmatrix} \cos b\cos l \\ \cos b\sin l \\ \sin b \end{pmatrix} = \mathbf{A}'_G \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \qquad [1.5.13]$$

The variation in $\mathbf{u}$ due to small changes in the spherical coordinates can be written, in analogy with Equation 1.2.14:

$$\Delta\mathbf{u} = [\,\mathbf{p}\ \mathbf{q}\,]\begin{pmatrix}\Delta\alpha*\\ \Delta\delta\end{pmatrix} = [\,\mathbf{p}_{\mathrm{K}}\ \mathbf{q}_{\mathrm{K}}\,]\begin{pmatrix}\Delta\lambda*\\ \Delta\beta\end{pmatrix} = [\,\mathbf{p}_{\mathrm{G}}\ \mathbf{q}_{\mathrm{G}}\,]\begin{pmatrix}\Delta l*\\ \Delta b\end{pmatrix} \qquad [1.5.14]$$

where $\Delta\lambda* = \Delta\lambda\cos\beta$, etc.; $\mathbf{p}_{\mathrm{K}}$ and $\mathbf{q}_{\mathrm{K}}$ are unit vectors perpendicular to $\mathbf{u}$ in the directions of increasing λ and β, respectively, with corresponding notations in the galactic system. These vectors are parts of the normal triads $[\,\mathbf{p}_{\mathrm{K}}\ \mathbf{q}_{\mathrm{K}}\ \mathbf{r}\,]$ and $[\,\mathbf{p}_{\mathrm{G}}\ \mathbf{q}_{\mathrm{G}}\ \mathbf{r}\,]$ relative to the ecliptic and galactic systems, defined at the reference direction $\mathbf{r} = \mathbf{u}$ in analogy with the normal triad relative to the equatorial system introduced in Section 1.2.8 (Equation 1.2.15). They may be computed from:

$$\begin{aligned}\mathbf{p} &= \langle \mathbf{z}\times\mathbf{r}\rangle & \mathbf{q} &= \mathbf{r}\times\mathbf{p}\\ \mathbf{p}_{\mathrm{K}} &= \langle \mathbf{z}_{\mathrm{K}}\times\mathbf{r}\rangle & \mathbf{q}_{\mathrm{K}} &= \mathbf{r}\times\mathbf{p}_{\mathrm{K}}\\ \mathbf{p}_{\mathrm{G}} &= \langle \mathbf{z}_{\mathrm{G}}\times\mathbf{r}\rangle & \mathbf{q}_{\mathrm{G}} &= \mathbf{r}\times\mathbf{p}_{\mathrm{G}}\end{aligned} \qquad [1.5.15]$$

[Note that, with respect to the equatorial system, the components of $\mathbf{z}$ are given by the column matrix $(0, 0, 1)'$, while the components of $\mathbf{z}_{\mathrm{K}}$ and $\mathbf{z}_{\mathrm{G}}$ are given by the third columns in the matrices $\mathbf{A}_{\mathrm{K}}$ and $\mathbf{A}_{\mathrm{G}}$, respectively. Thus, the vector operations in Equation 1.5.15 could all be carried out in the equatorial system by means of standard operations on the corresponding column matrices.]

Using the orthonormality of the tangent vectors it follows from Equation 1.5.14 that:

$$\begin{pmatrix}\Delta\lambda*\\ \Delta\beta\end{pmatrix} = \begin{pmatrix}\mathbf{p}_{\mathrm{K}}'\mathbf{p} & \mathbf{p}_{\mathrm{K}}'\mathbf{q}\\ \mathbf{q}_{\mathrm{K}}'\mathbf{p} & \mathbf{q}_{\mathrm{K}}'\mathbf{q}\end{pmatrix}\begin{pmatrix}\Delta\alpha*\\ \Delta\delta\end{pmatrix} \qquad [1.5.16]$$

and:

$$\begin{pmatrix}\Delta l*\\ \Delta b\end{pmatrix} = \begin{pmatrix}\mathbf{p}_{\mathrm{G}}'\mathbf{p} & \mathbf{p}_{\mathrm{G}}'\mathbf{q}\\ \mathbf{q}_{\mathrm{G}}'\mathbf{p} & \mathbf{q}_{\mathrm{G}}'\mathbf{q}\end{pmatrix}\begin{pmatrix}\Delta\alpha*\\ \Delta\delta\end{pmatrix} \qquad [1.5.17]$$

The 2×2 matrices in the last two equations contain the partial derivatives of the ecliptic and galactic coordinates with respect to the equatorial ones. Computationally, they are obtained as scalar products of the relevant tangential vectors expressed in any convenient coordinate system, such as the equatorial. Considering the proper motion components as the time derivatives of $\Delta\alpha*$, $\Delta\delta$, etc. (with the tangent vectors regarded as fixed), their transformations are similarly given by:

$$\begin{pmatrix}\mu_{\lambda*}\\ \mu_{\beta}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_{\mathrm{K}}'\mathbf{p} & \mathbf{p}_{\mathrm{K}}'\mathbf{q}\\ \mathbf{q}_{\mathrm{K}}'\mathbf{p} & \mathbf{q}_{\mathrm{K}}'\mathbf{q}\end{pmatrix}\begin{pmatrix}\mu_{\alpha*}\\ \mu_{\delta}\end{pmatrix} \qquad [1.5.18]$$

and:

$$\begin{pmatrix}\mu_{l*}\\ \mu_{b}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_{\mathrm{G}}'\mathbf{p} & \mathbf{p}_{\mathrm{G}}'\mathbf{q}\\ \mathbf{q}_{\mathrm{G}}'\mathbf{p} & \mathbf{q}_{\mathrm{G}}'\mathbf{q}\end{pmatrix}\begin{pmatrix}\mu_{\alpha*}\\ \mu_{\delta}\end{pmatrix} \qquad [1.5.19]$$

The complete transformation of the equatorial position and proper motion into the ecliptic system is given by Equations 1.5.12 and 1.5.18; the parallax is of course independent of the coordinate system. Taking the five astrometric parameters in the standard order of Equation 1.2.4, the Jacobian matrix for the transformation is:

$$\mathbf{J} = \begin{pmatrix} c & s & 0 & 0 & 0\\ -s & c & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & c & s\\ 0 & 0 & 0 & -s & c\end{pmatrix} \qquad [1.5.20]$$

with $c = \mathbf{p}_{\mathrm{K}}'\mathbf{p} = \mathbf{q}_{\mathrm{K}}'\mathbf{q}$ and $s = \mathbf{p}_{\mathrm{K}}'\mathbf{q} = -\mathbf{q}_{\mathrm{K}}'\mathbf{p}$. The transformation of the equatorial parameters into the galactic system is similarly given by Equations 1.5.13 and 1.5.19; the Jacobian matrix is again given by Equation 1.5.20, now with $c = \mathbf{p}_{\mathrm{G}}'\mathbf{p} = \mathbf{q}_{\mathrm{G}}'\mathbf{q}$ and $s = \mathbf{p}_{\mathrm{G}}'\mathbf{q} = -\mathbf{q}_{\mathrm{G}}'\mathbf{p}$.

1.5.4. Epoch Transformation: Simplified Treatment

The simplistic formulae for transforming a celestial position (α, δ) from the catalogue epoch T_0 to the arbitrary epoch T are:

$$\begin{aligned} \alpha &= \alpha_0 + (T - T_0)\,\mu_{\alpha*0} \sec\delta_0 \\ \delta &= \delta_0 + (T - T_0)\,\mu_{\delta 0} \end{aligned} \qquad [1.5.21]$$

(where the $\sec\delta_0$ factor compensates the $\cos\delta_0$ factor implicit in $\mu_{\alpha*}$). This is not a good physical model of how the stars move on the sky: in general it describes a curved, spiralling motion towards one of the poles, whereas (unperturbed) real stars are expected to move along great-circle arcs. Although the difference with respect to a rigorous model (Section 1.5.5) is usually very small, it may become significant in special cases, in particular for stars near the celestial poles or when propagating over very long time intervals. Equation 1.5.21 should therefore not be used in software intended for general application. Nevertheless, it provides in many cases a useful first-order approximation suitable for hand calculation and for estimating the positional uncertainties at arbitrary epochs.

In this simplified model the slow changes in the proper motion components and in the parallax are neglected and the Jacobian matrix for the epoch transformation is then:

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad [1.5.22]$$

with $t = T - T_0$. The covariance matrix for the five astrometric parameters at epoch T are obtained from Equation 1.5.4; this gives in particular for the variances in position:

$$\begin{aligned} \sigma_{\alpha*}^2 &= \left[\sigma_{\alpha*}^2 + 2t\rho_{\alpha*}^{\mu_{\alpha*}}\sigma_{\alpha*}\sigma_{\mu_{\alpha*}} + t^2\sigma_{\mu_{\alpha*}}^2\right]_0 \\ \sigma_{\delta}^2 &= \left[\sigma_{\delta}^2 + 2t\rho_{\delta}^{\mu_\delta}\sigma_{\delta}\sigma_{\mu_\delta} + t^2\sigma_{\mu_\delta}^2\right]_0 \end{aligned} \qquad [1.5.23]$$

(with all quantities in the right members referring to epoch T_0). Minimising these variances with respect to t results in the previously given expressions for the mean observational epochs in right ascension and declination; see Equations 1.2.6–1.2.10.

1.5.5. Epoch Transformation: Rigorous Treatment

In this section the rigorous transformation of parameters and covariances is formulated, based on the standard model of stellar motion described in Section 1.2.8. This is the epoch transformation incorporated in the *Celestia 2000* software. Because of its relative complexity, Fortran and C implementations of the transformation are included on ASCII CD-ROM disc 1 (see Section 2.11).

The six-dimensional parameter vector a: Recall that the standard model assumes uniform space velocity for the object: its path on the celestial sphere (as seen from the solar system barycentre, i.e. without the displacement due to parallax) is a great-circle arc. The angular velocity (proper motion) along this arc is variable, reaching a maximum when the object is at the point closest to the Sun along its rectilinear path; the distance (parallax) and distance rate (radial velocity) are also variable. In the rigorous treatment the variation of all *six* parameters α, δ, π, $\mu_{\alpha*}$, μ_δ, V_R must therefore be considered, and the parameter vector $\mathbf{a}$ is now six-dimensional, associated with the 6×6 covariance matrix $\mathbf{C}$. For computational reasons the sixth parameter is taken to be the radial velocity divided by distance, or more precisely:

$$\zeta = V_R\pi/A \qquad [1.5.24]$$

where A is the astronomical unit. ζ is conveniently measured in mas/yr, which is obtained by expressing V_R in km/s and A in km yr/s (= A_v in Table 1.2.2; see also the note after Equation 1.2.17). Its value as computed from the catalogue (referring to epoch T_0) is denoted ζ_0. The use of ζ instead of V_R as the sixth parameter allows the case of measured $\pi \le 0$ to be handled gracefully (if unphysically) by the general algorithm.

Propagation of a: The propagation of the barycentric direction, and hence of α and δ, is given by Equation 1.2.16. Using notations from Section 1.2.8 and introducing the distance factor:

$$f = |\mathbf{b}(0)|\,|\mathbf{b}(t)|^{-1} = [1 + 2\zeta_0 t + (\mu_0^2 + \zeta_0^2)t^2]^{-1/2} \qquad [1.5.25]$$

(where $\mu_0^2 = |\boldsymbol{\mu}_0|^2 = \mu_{\alpha*0}^2 + \mu_{\delta0}^2$), the propagation of the barycentric direction is:

$$\mathbf{u} = [\mathbf{r}_0(1 + \zeta_0 t) + \boldsymbol{\mu}_0 t]f \qquad [1.5.26]$$

and the propagation of the parallax becomes:

$$\pi = \pi_0 f \qquad [1.5.27]$$

Direct differentiation of Equation 1.5.26 now gives the propagated proper motion vector:

$$\boldsymbol{\mu} = \frac{d\mathbf{u}}{dt} = \left[\boldsymbol{\mu}_0(1 + \zeta_0 t) - \mathbf{r}_0\mu_0^2 t\right] f^3 \qquad [1.5.28]$$

while the propagated sixth parameter is found to be:

$$\zeta = \frac{db}{dt}\frac{\pi}{A} = \left[\zeta_0 + (\mu_0^2 + \zeta_0^2)t\right] f^2 \qquad [1.5.29]$$

The barycentric direction and proper motion vector, taken at the catalogue epoch, are:

$$\mathbf{u}_0 = \mathbf{r}_0 \qquad [1.5.30]$$

$$\boldsymbol{\mu}_0 = \mathbf{p}_0\mu_{\alpha*0} + \mathbf{q}_0\mu_{\delta0} \qquad [1.5.31]$$

where [$\mathbf{p}_0$ $\mathbf{q}_0$ $\mathbf{r}_0$] is the normal triad introduced in Section 1.2.8; its equatorial components are given by Equations 1.2.11 and 1.2.15.

The celestial coordinates (α, δ) at epoch T are obtained from $\mathbf{u}$ in the usual manner, using Equation 1.5.5. However, to obtain the proper motion components $(\mu_{\alpha*}, \mu_\delta)$ from the vector $\boldsymbol{\mu}$ it is necessary to resolve the latter along the tangential vectors $\mathbf{p}$ and $\mathbf{q}$ *at the propagated position*, i.e. with the tangent point located at $\mathbf{u}$ and with $\mathbf{p}$ pointing towards the local north. The proper motion components are thus:

$$\mu_{\alpha*} = \mathbf{p}'\boldsymbol{\mu} \qquad \mu_\delta = \mathbf{q}'\boldsymbol{\mu} \qquad [1.5.32]$$

with $\mathbf{p}$ and $\mathbf{q}$ defined in terms of $\mathbf{u}$ or (α, δ) (at epoch T) according to Equation 1.2.15 or 1.5.15.

The above formulae describe the transformation of (α_0, δ_0, π_0, $\mu_{\alpha*0}$, $\mu_{\delta0}$, V_{R0}) at epoch T_0 into (α, δ, π, $\mu_{\alpha*}$, μ_δ, V_R) at epoch T. This transformation is rigorously reversible, so that a second transformation from T to T_0 recovers the original six parameters.

Propagation of C: The transformation of covariances requires the calculation of all 36 partial derivatives constituting the Jacobian matrix in $\mathbf{C} = \mathbf{J}\mathbf{C}_0\mathbf{J}'$. This is relatively straightforward, if somewhat tedious, and the complete expressions are given below. However, one further consideration is necessary. The proper motion components depend on the tangential vectors $\mathbf{p}$ and $\mathbf{q}$ (Equation 1.5.32), and these in turn depend on the reference direction $\mathbf{r}$ set equal to the propagated position $\mathbf{u}$ (Equation 1.5.15). The question arises whether the normal triad $[\,\mathbf{p}\;\mathbf{q}\;\mathbf{r}\,]$ should be regarded as a fixed, error-free reference frame for the proper motion vector, or whether the uncertainties in $\mathbf{u}$ should be propagated through the definition of this triad, and subsequently to the proper motion components.

It appears that either option could be adopted, as long as it is consistently applied, and that the choice between them is a matter of convention. The reduction procedures used to construct the Hipparcos and Tycho Catalogues implicitly adopt the first option, which also seems closer to an intuitive understanding of the uncertainty of the proper motion (as representing an uncertainty in the physical vector $\boldsymbol{\mu}$). Consequently the same convention has been adopted for the calculation of the Jacobian matrix below. The practical consequence is that the two normal triads $[\,\mathbf{p}_0\;\mathbf{q}_0\;\mathbf{r}_0\,]$ and $[\,\mathbf{p}\;\mathbf{q}\;\mathbf{r}\,]$ must be regarded as fixed in all the calculations, which also motivates the somewhat formal distinction between the vectors $\mathbf{r}_0$ and $\mathbf{u}_0$, and between $\mathbf{r}$ and $\mathbf{u}$.

The elements of the Jacobian matrix are given hereafter:

$$[\mathbf{J}]_{11} = \frac{\partial\alpha*}{\partial\alpha*_0} = \mathbf{p}'\mathbf{p}_0(1+\zeta_0 t)f - \mathbf{p}'\mathbf{r}_0\mu_{\alpha*0}tf \qquad [1.5.33]$$

$$[\mathbf{J}]_{12} = \frac{\partial\alpha*}{\partial\delta_0} = \mathbf{p}'\mathbf{q}_0(1+\zeta_0 t)f - \mathbf{p}'\mathbf{r}_0\mu_{\delta 0}tf \qquad [1.5.34]$$

$$[\mathbf{J}]_{13} = \frac{\partial\alpha*}{\partial\pi_0} = 0 \qquad [1.5.35]$$

$$[\mathbf{J}]_{14} = \frac{\partial\alpha*}{\partial\mu_{\alpha*0}} = \mathbf{p}'\mathbf{p}_0 tf \qquad [1.5.36]$$

$$[\mathbf{J}]_{15} = \frac{\partial\alpha*}{\partial\mu_{\delta 0}} = \mathbf{p}'\mathbf{q}_0 tf \qquad [1.5.37]$$

$$[\mathbf{J}]_{16} = \frac{\partial\alpha*}{\partial\zeta_0} = -\mu_{\alpha*}t^2 \qquad [1.5.38]$$

$$[\mathbf{J}]_{21} = \frac{\partial\delta}{\partial\alpha*_0} = \mathbf{q}'\mathbf{p}_0(1+\zeta_0 t)f - \mathbf{q}'\mathbf{r}_0\mu_{\alpha*0}tf \qquad [1.5.39]$$

$$[\mathbf{J}]_{22} = \frac{\partial\delta}{\partial\delta_0} = \mathbf{q}'\mathbf{q}_0(1+\zeta_0 t)f - \mathbf{q}'\mathbf{r}_0\mu_{\delta 0}tf \qquad [1.5.40]$$

$$[\mathbf{J}]_{23} = \frac{\partial\delta}{\partial\pi_0} = 0 \qquad [1.5.41]$$

$$[\mathbf{J}]_{24} = \frac{\partial\delta}{\partial\mu_{\alpha*0}} = \mathbf{q}'\mathbf{p}_0 tf \qquad [1.5.42]$$

$$[\mathbf{J}]_{25} = \frac{\partial\delta}{\partial\mu_{\delta 0}} = \mathbf{q}'\mathbf{q}_0 tf \qquad [1.5.43]$$

$$[\mathbf{J}]_{26} = \frac{\partial\delta}{\partial\zeta_0} = -\mu_{\delta}t^2 \qquad [1.5.44]$$

$$[\mathbf{J}]_{31} = \frac{\partial\pi}{\partial\alpha*_0} = 0 \qquad [1.5.45]$$

$$[\mathbf{J}]_{32} = \frac{\partial \pi}{\partial \delta_0} = 0 \qquad [1.5.46]$$

$$[\mathbf{J}]_{33} = \frac{\partial \pi}{\partial \pi_0} = f \qquad [1.5.47]$$

$$[\mathbf{J}]_{34} = \frac{\partial \pi}{\partial \mu_{\alpha*0}} = -\pi \mu_{\alpha*0} t^2 f^2 \qquad [1.5.48]$$

$$[\mathbf{J}]_{35} = \frac{\partial \pi}{\partial \mu_{\delta 0}} = -\pi \mu_{\delta 0} t^2 f^2 \qquad [1.5.49]$$

$$[\mathbf{J}]_{36} = \frac{\partial \pi}{\partial \zeta_0} = -\pi(1+\zeta_0 t) t f^2 \qquad [1.5.50]$$

$$[\mathbf{J}]_{41} = \frac{\partial \mu_{\alpha*}}{\partial \alpha*_0} = -\mathbf{p}'\mathbf{p}_0 \mu_0^2 t f^3 - \mathbf{p}'\mathbf{r}_0 \mu_{\alpha*0}(1+\zeta_0 t) f^3 \qquad [1.5.51]$$

$$[\mathbf{J}]_{42} = \frac{\partial \mu_{\alpha*}}{\partial \delta_0} = -\mathbf{p}'\mathbf{q}_0 \mu_0^2 t f^3 - \mathbf{p}'\mathbf{r}_0 \mu_{\delta 0}(1+\zeta_0 t) f^3 \qquad [1.5.52]$$

$$[\mathbf{J}]_{43} = \frac{\partial \mu_{\alpha*}}{\partial \pi_0} = 0 \qquad [1.5.53]$$

$$[\mathbf{J}]_{44} = \frac{\partial \mu_{\alpha*}}{\partial \mu_{\alpha*0}} = \mathbf{p}'\mathbf{p}_0(1+\zeta_0 t) f^3 - 2\mathbf{p}'\mathbf{r}_0 \mu_{\alpha*0} t f^3 - 3\mu_{\alpha*}\mu_{\alpha*0} t^2 f^2 \qquad [1.5.54]$$

$$[\mathbf{J}]_{45} = \frac{\partial \mu_{\alpha*}}{\partial \mu_{\delta 0}} = \mathbf{p}'\mathbf{q}_0(1+\zeta_0 t) f^3 - 2\mathbf{p}'\mathbf{r}_0 \mu_{\delta 0} t f^3 - 3\mu_{\alpha*}\mu_{\delta 0} t^2 f^2 \qquad [1.5.55]$$

$$[\mathbf{J}]_{46} = \frac{\partial \mu_{\alpha*}}{\partial \zeta_0} = \mathbf{p}'[\boldsymbol{\mu}_0 f - 3\boldsymbol{\mu}(1+\zeta_0 t)] t f^2 \qquad [1.5.56]$$

$$[\mathbf{J}]_{51} = \frac{\partial \mu_{\delta}}{\partial \alpha*_0} = -\mathbf{q}'\mathbf{p}_0 \mu_0^2 t f^3 - \mathbf{q}'\mathbf{r}_0 \mu_{\alpha*0}(1+\zeta_0 t) f^3 \qquad [1.5.57]$$

$$[\mathbf{J}]_{52} = \frac{\partial \mu_{\delta}}{\partial \delta_0} = -\mathbf{q}'\mathbf{q}_0 \mu_0^2 t f^3 - \mathbf{q}'\mathbf{r}_0 \mu_{\delta 0}(1+\zeta_0 t) f^3 \qquad [1.5.58]$$

$$[\mathbf{J}]_{53} = \frac{\partial \mu_{\delta}}{\partial \pi_0} = 0 \qquad [1.5.59]$$

$$[\mathbf{J}]_{54} = \frac{\partial \mu_{\delta}}{\partial \mu_{\alpha*0}} = \mathbf{q}'\mathbf{p}_0(1+\zeta_0 t) f^3 - 2\mathbf{q}'\mathbf{r}_0 \mu_{\alpha*0} t f^3 - 3\mu_{\delta}\mu_{\alpha*0} t^2 f^2 \qquad [1.5.60]$$

$$[\mathbf{J}]_{55} = \frac{\partial \mu_{\delta}}{\partial \mu_{\delta 0}} = \mathbf{q}'\mathbf{q}_0(1+\zeta_0 t) f^3 - 2\mathbf{q}'\mathbf{r}_0 \mu_{\delta 0} t f^3 - 3\mu_{\delta}\mu_{\delta 0} t^2 f^2 \qquad [1.5.61]$$

$$[\mathbf{J}]_{56} = \frac{\partial \mu_{\delta}}{\partial \zeta_0} = \mathbf{q}'[\boldsymbol{\mu}_0 f - 3\boldsymbol{\mu}(1+\zeta_0 t)] t f^2 \qquad [1.5.62]$$

$$[\mathbf{J}]_{61} = \frac{\partial \zeta}{\partial \alpha*_0} = 0 \qquad [1.5.63]$$

$$[\mathbf{J}]_{62} = \frac{\partial \zeta}{\partial \delta_0} = 0 \qquad [1.5.64]$$

$$[\mathbf{J}]_{63} = \frac{\partial \zeta}{\partial \pi_0} = 0 \qquad [1.5.65]$$

$$[\mathbf{J}]_{64} = \frac{\partial \zeta}{\partial \mu_{\alpha*0}} = 2\mu_{\alpha*0}(1+\zeta_0 t) t f^4 \qquad [1.5.66]$$

$$[\mathbf{J}]_{65} = \frac{\partial \zeta}{\partial \mu_{\delta 0}} = 2\mu_{\delta 0}(1+\zeta_0 t) t f^4 \qquad [1.5.67]$$

$$[\mathbf{J}]_{66} = \frac{\partial \zeta}{\partial \zeta_0} = [(1+\zeta_0 t)^2 - \mu_0^2 t^2] f^4 \qquad [1.5.68]$$

Initialization of $\mathbf{C}_0$: The Hipparcos and Tycho Catalogues provide the first five rows and columns of elements in $\mathbf{C}_0$ according to Equation 1.2.5. For the rigorous propagation this must be augmented with a sixth row and column related to the parameter $\zeta_0 = V_{R0}\pi_0/A$. If the (spectroscopic) radial velocity V_{R0} has the standard error $\sigma_{V_{R0}}$, and is assumed to be statistically independent of the astrometric parameters in the catalogue, then the required additional elements in $\mathbf{C}_0$ are:

$$\left.\begin{aligned} [\mathbf{C}_0]_{i6} = [\mathbf{C}_0]_{6i} &= (V_{R0}/A)\,[\mathbf{C}_0]_{i3}, \quad i = 1 \ldots 5 \\ [\mathbf{C}_0]_{66} &= (V_{R0}/A)^2\,[\mathbf{C}_0]_{33} + (\pi_0/A)^2\,\sigma^2_{V_{R0}} \end{aligned}\right\} \qquad [1.5.69]$$

If the radial velocity is not known, ζ_0 should be set to zero and the corresponding elements in $\mathbf{C}_0$ could also be zeroed. However, it could also be argued that $\sigma_{V_{R0}}$ should be set to the expected velocity dispersion of the stellar type in question, in which case $[\mathbf{C}_0]_{66}$ would in general still be positive according to Equation 1.5.69. This means that the unknown perspective acceleration is accounted for in the uncertainty of the propagated astrometric parameters. In the *Celestia 2000* software, $V_{R0} = 0$ is assumed for all stars except the 21 listed in Table 1.2.3, and $\sigma_{V_{R0}} = 0$ is assumed for all stars.

It should be noted that strict reversal of the transformation (from T to T_0), according to the standard model of stellar motion, is only possible if the full six-dimensional parameter vector and covariance is considered.

1.5.6. Calculation of Space Coordinates and Velocity

The calculation of the rectangular space coordinates and velocity of a star from the astrometric parameters, supplemented by the radial velocity, was considered already in Section 1.2.8, as part of the standard model of stellar motion. For convenience, the relevant formulae are summarized hereafter. All quantities may be taken to refer to the catalogue epoch J1991.25, although results for other epochs can be obtained by applying first the transformations in the preceding sections (for brevity, the subscript $_0$ is subsequently dropped).

Let $\mathbf{b}$ be the barycentric position of the star, measured in parsec, and $\mathbf{v}$ its barycentric space velocity, measured in km/s. According to Section 1.2.8, these vectors can be written:

$$\mathbf{b} = A_p \mathbf{u}/\pi \qquad [1.5.70]$$

where $\mathbf{u}$ is the direction defined by Equation 1.2.16, and:

$$\mathbf{v} = (\mathbf{p}\mu_{\alpha*}A_v/\pi + \mathbf{q}\mu_\delta A_v/\pi + \mathbf{r}V_R)k \qquad [1.5.71]$$

Here, $[\,\mathbf{p}\ \mathbf{q}\ \mathbf{r}\,]$ is the normal triad introduced in Section 1.2.8, with $\mathbf{r} = \mathbf{u}$. $A_p =$ 1000 mas pc and $A_v = 4.74047...$ km yr s^{-1} designate the astronomical unit expressed in the appropriate units (Table 1.2.2), and π, $\mu_{\alpha*}$ and μ_δ are the parallax and proper motion components in mas and mas/yr. $k = (1 - V_R/c)^{-1}$ is the Doppler factor explained in connection with Equation 1.2.21.

In the equatorial system the components of the normal triad $[\,\mathbf{p}\ \mathbf{q}\ \mathbf{r}\,]$ are given by the matrix:

$$\mathbf{R} = \begin{pmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{pmatrix} = \begin{pmatrix} -\sin\alpha & -\sin\delta\cos\alpha & \cos\delta\cos\alpha \\ \cos\alpha & -\sin\delta\sin\alpha & \cos\delta\sin\alpha \\ 0 & \cos\delta & \sin\delta \end{pmatrix} \qquad [1.5.72]$$

(cf. Equations 1.2.11 and 1.2.15). The equatorial components of **b** and **v** may thus be written in matrix form as:

$$\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} 0 \\ 0 \\ A_p/\pi \end{pmatrix} \qquad [1.5.73]$$

and:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} k\mu_{\alpha*}A_v/\pi \\ k\mu_\delta A_v/\pi \\ kV_\mathrm{R} \end{pmatrix} \qquad [1.5.74]$$

The galactic components of **b** and **v** are obtained through pre-multiplication by $\mathbf{A}'_\mathrm{G}$, as in Equation 1.5.13.

For kinematical applications the covariance of the space velocity components may be required. More generally, the 6×6 covariance matrix of the position-velocity vector $\mathbf{s} \equiv (b_x, b_y, b_z, v_x, v_y, v_z)'$ may be calculated (again in equatorial coordinates) as:

$$\mathrm{Cov}(\mathbf{s}) = \mathbf{J} \begin{pmatrix} \mathbf{C} & \mathbf{0}_{51} \\ \mathbf{0}_{15} & \sigma^2_{V_\mathrm{R}} \end{pmatrix} \mathbf{J}' \qquad [1.5.75]$$

where **C** is the 5×5 covariance matrix of the astrometric parameters, $\mathbf{0}_{mn}$ is the $m \times n$ matrix of zeroes, σ_{V_R} the standard error of the radial velocity, and:

$$\mathbf{J} = \begin{pmatrix} A_p p_x/\pi & A_p q_x/\pi & -A_p r_x/\pi^2 & 0 & 0 & 0 \\ A_p p_y/\pi & A_p q_y/\pi & -A_p r_y/\pi^2 & 0 & 0 & 0 \\ A_p p_z/\pi & A_p q_z/\pi & -A_p r_z/\pi^2 & 0 & 0 & 0 \\ 0 & 0 & -(p_x\mu_{\alpha*} + q_x\mu_\delta)A_v/\pi^2 & p_x A_v/\pi & q_x A_v/\pi & r_x \\ 0 & 0 & -(p_y\mu_{\alpha*} + q_y\mu_\delta)A_v/\pi^2 & p_y A_v/\pi & q_y A_v/\pi & r_y \\ 0 & 0 & -(p_z\mu_{\alpha*} + q_z\mu_\delta)A_v/\pi^2 & p_z A_v/\pi & q_z A_v/\pi & r_z \end{pmatrix} \qquad [1.5.76]$$

is the Jacobian matrix of the transformation from $(\alpha*, \delta, \pi, \mu_{\alpha*}, \mu_\delta, V_\mathrm{R})'$ to **s**. (The Doppler factor k has been neglected for the calculation of the partial derivatives.) The covariance of the space velocity is the lower-right 3×3 submatrix of Equation 1.5.75.

In the galactic system the components of the position-velocity vector are given by the 6×1 matrix **Gs**, where:

$$\mathbf{G} = \begin{pmatrix} \mathbf{A}'_\mathrm{G} & \mathbf{0}_{33} \\ \mathbf{0}_{33} & \mathbf{A}'_\mathrm{G} \end{pmatrix} \qquad [1.5.77]$$

according to Equation 1.5.13; the associated covariance is therefore:

$$\mathrm{Cov}(\mathbf{Gs}) = \mathbf{GJ} \begin{pmatrix} \mathbf{C} & \mathbf{0}_{51} \\ \mathbf{0}_{15} & \sigma^2_{V_\mathrm{R}} \end{pmatrix} \mathbf{J}'\mathbf{G}' \qquad [1.5.78]$$

1.5.7. Relation to the J2000(FK5) Reference Frame

As explained in Section 1.2.2, the Hipparcos and Tycho Catalogues materialise the International Celestial Reference System by defining a reference frame, which may be designated ICRS(Hipparcos). This supersedes the optical reference frame defined by the FK5 catalogue, which was formally based on the mean equator and dynamical equinox of J2000 and therefore properly designated J2000(FK5).

Since all the stars in the basic FK5 catalogue are also contained in the Hipparcos Catalogue, the relationship between the two reference frames is open to investigation by direct comparison of the positions and proper motions in the two catalogues. The results of one such comparison are given in Volume 3.

A complete characterisation of the relation between J2000(FK5) and ICRS(Hipparcos) is difficult to achieve, given the relatively small number of stars defining the FK5 reference frame, and the intricate and possibly colour- and magnitude-dependent pattern of systematic (in particular zonal) differences, combined with the perturbing effects on the proper motions of undetected astrometric binaries. Nevertheless, certain relations on a global scale may be established with relative ease, in particular the most fundamental one corresponding to a difference in the mean orientation and spin between the two catalogues.

At the epoch T_0 = J1991.25 let $(\alpha_\mathrm{F}, \delta_\mathrm{F})$ be the barycentric coordinates of an object in the J2000(FK5) frame, and $(\alpha_\mathrm{H}, \delta_\mathrm{H})$ its coordinates in the ICRS(Hipparcos) frame. If the two frames are related by a pure rigid-body rotation, the coordinate differences can be written, in the small-angle approximation:

$$\begin{aligned}(\alpha_\mathrm{F}-\alpha_\mathrm{H})\cos\delta &= -\varepsilon_{0x}\,\sin\delta\cos\alpha-\varepsilon_{0y}\,\sin\delta\sin\alpha+\varepsilon_{0z}\,\cos\delta\\ \delta_\mathrm{F}-\delta_\mathrm{H} &= +\varepsilon_{0x}\,\sin\alpha-\varepsilon_{0y}\,\cos\alpha\end{aligned} \qquad [1.5.79]$$

(either set of coordinates may be used in the trigonometric factors). The vector $\boldsymbol{\varepsilon}_0 = (\varepsilon_{0x}\ \varepsilon_{0y}\ \varepsilon_{0z})'$ represents the orientation difference between the frames, taken in the sense F−H, at epoch T_0 (L. Lindegren & J. Kovalevsky, 1995, *Astronomy & Astrophysics*, 304, 189). The differences in proper motion, to the extent they are described by a pure spin, can similarly be written:

$$\begin{aligned}(\mu_{\alpha*})_\mathrm{F}-(\mu_{\alpha*})_\mathrm{H} &= -\omega_x\,\sin\delta\cos\alpha-\omega_y\,\sin\delta\sin\alpha+\omega_z\cos\delta\\ (\mu_\delta)_\mathrm{F}-(\mu_\delta)_\mathrm{H} &= +\omega_x\,\sin\alpha-\omega_y\,\cos\alpha\end{aligned} \qquad [1.5.80]$$

where the vector $\boldsymbol{\omega} = (\omega_x\ \omega_y\ \omega_z)'$ represents the spin difference between the frames, again taken in the sense F−H.

However, fitting the rotational parameters $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\omega}$ directly to the catalogue differences, by means of the above equations, results in a solution which is significantly affected by the non-zero projection of the largest zonal differences onto the rotational terms. For this reason a more general representation was preferred, in which the vector fields $[(\alpha_\mathrm{F}-\alpha_\mathrm{H})\cos\delta,\ \delta_\mathrm{F}-\delta_\mathrm{H}]$ and $[(\mu_{\alpha*})_\mathrm{F}-(\mu_{\alpha*})_\mathrm{H},\ (\mu_\delta)_\mathrm{F}-(\mu_\delta)_\mathrm{H}]$ are decomposed on a set of orthogonal vectorial harmonics. The first degree of these harmonics represents the pure rotation while the harmonics of higher degree account for the zonal differences. The root mean square of the residuals after removal of all significant terms allows the accuracy of each Fourier component of the decomposition to be estimated.

Applying this method to the estimation of the orientation and spin differences between the FK5 and the Hipparcos Catalogue gives:

$$\begin{aligned}
\epsilon_{0x} &= -18.8 \pm 2.3 \text{ mas} \\
\epsilon_{0y} &= -12.3 \pm 2.3 \text{ mas} \\
\epsilon_{0z} &= +16.8 \pm 2.3 \text{ mas} \\
\omega_x &= -0.10 \pm 0.10 \text{ mas yr}^{-1} \\
\omega_y &= +0.43 \pm 0.10 \text{ mas yr}^{-1} \\
\omega_z &= +0.88 \pm 0.10 \text{ mas yr}^{-1}
\end{aligned} \qquad [1.5.81]$$

where the orientation parameters refer to the epoch J1991.25. This preliminary result was based on the catalogue differences for all 1535 FK5 stars without filtering; no star was removed in the comparison. The remaining differences once the rotation has been applied may be as large as 150 mas, because of the large zonal differences which show up in the harmonics of higher degree.

A similar decomposition based on only 1232 FK5 stars, after the double stars and the suspected astrometric binaries had been excluded, led to very similar values for the orientation and spin. Likewise various binnings in cells of 100, 200 or 400 square degrees was attempted and found to give comparable results. Selecting stars according to their brightness produced different solutions for the rotation and spin parameters slightly outside the above standard errors, demonstrating the difficulty of establishing a well-defined relation between J2000(FK5) and ICRS(Hipparcos).