# 11. SPHERE SOLUTION

*The sphere solution combined the star abscissae obtained in the great-circle reductions (Chapter 9) into the positions, parallaxes and proper motions of the stars, expressed in a globally coherent coordinate system. It consisted of two processes: (1) the determination of the abscissa zero points of all the reference great circles, which was the sphere solution proper; and (2) the determination of the astrometric parameters of individual objects. While the first process required a simultaneous least-squares solution of a large number of stars, which must all be consistent with the single-star model, the second process could be performed sequentially using several different models as appropriate for each object. In this chapter the basic observation equation is derived and the numerical methods of solution used by FAST and NDAC are outlined. The final section of the chapter deals with the 'rank deficiency problem' and reports some numerical experiments to study this problem.*

## 11.1. Introduction

The purpose of the sphere solution was to calculate, from the abscissae determined by the great-circle reductions, the astrometric parameters of the stars: both components of position, both components of the proper motion, and the parallax. This chapter provides a general formulation of this process. In practice the successive sphere solutions performed by the FAST and NDAC consortia differed in many details, especially concerning the use of 'global' parameters for the modelling of instrument chromaticity and the harmonic components of the abscissa errors; these detailed aspects as well as the numerical characteristics of the successive solutions are covered in Chapter 16.

In order to take advantage of the symmetry of the nominal scanning law with respect to the ecliptic, all computations in the FAST Consortium were made in ecliptic coordinates. In the NDAC Consortium, equatorial coordinates were used throughout. This difference is immaterial for a general exposition of the sphere solution and largely disappears when vector algebra is used in its formulation. When a reference to the celestial coordinates is nevertheless needed, ecliptic coordinates ($\lambda$, $\beta$) will be used, and the ecliptic is taken to be the fundamental plane. To obtain the corresponding equations and conventions according to NDAC it is only necessary to substitute ($\alpha$, $\delta$) and the equator. The generic celestial triad [ $\mathbf{x}\ \mathbf{y}\ \mathbf{z}$ ] may thus be taken to mean either the ecliptic or equatorial triad. The transformation between these two systems is completely defined by the value of the obliquity of the ecliptic ($\epsilon$), for which the IAU (1976) value at epoch J2000 was adopted (see Table 12.1).

The great-circle reductions determined the one-dimensional coordinates, or abscissae, of the stars along a number of different reference great circles ($j$). In an absolute sense, the abscissa is defined as the angle $v$, as seen from the designated pole of the reference great circle, from the ascending node of the reference great circle on the ecliptic to the topocentric coordinate direction of the object (Figure 11.1). It should be noted that the abscissa, being defined in terms of the coordinate direction of the object, is not affected by gravitational light deflection and stellar aberration; these effects, whose computation does not require an accurate astrometric knowledge of the object, were removed in the great-circle reductions.

In principle, therefore, the astrometric parameters of a given star $i$ can be computed on the basis of a geometrical model of its motion, using the abscissa values $v_{ji}$ as 'observations'. The only additional data required are the times of observation ($t_{ji}$), the corresponding reference great-circle poles ($\mathbf{R}_j$), and the barycentric locations of the satellite ($\mathbf{b}_j$). This process, known as the 'determination of astrometric parameters', can clearly be made on a star-by-star basis. However, it requires that the abscissae are actually available in the form described above, i.e. as the absolute angles from the ecliptic to the object, as measured on a number of great circles.

In reality the abscissae obtained in the great-circle reductions do not satisfy this condition. The main problem is the arbitrary origin of the abscissae introduced in each great-circle reduction. This means that the abscissae on a given reference great circle are measured, not from the ecliptic, but from some other, in principle unknown origin. Consequently a set of corrections $c_j$ need to be added in order to convert the abscissae into the absolute quantities required for the determination of astrometric parameters. These corrections can only be determined by simultaneously considering a large number of stars, and explicitly using the circumstance that the same correction applies to *all* the abscissae on the same reference great circle. However, even in this process, the corrections $c_j$ can only be determined in such a way that the corrected abscissae express the angles from a certain fundamental plane, which need not be exactly the ecliptic, nor even fixed with respect to the ecliptic; thus a basic indeterminacy of the celestial reference frame remains after the sphere solution.

The need to determine the abscissa origins is however not the only reason for doing a 'sphere solution' in which the measurements of a large number of stars scattered over the whole sphere are considered in a single solution. There are other, more subtle effects causing systematic shifts in the abscissae which cannot be eliminated in the great-circle reductions, but may be determined in the sphere solution, due to the additional constraints introduced by the stellar astrometric model. These effects include the component of instrument chromaticity that is constant in both fields of view, representing a colour dependence of the zero points $c_j$. Furthermore, some harmonic components of the abscissa error, notably the sixth harmonic, are more accurately estimated in the sphere solution than in the great-circle reductions. By including such additional unknowns in the sphere solution, their effects on the 'observed' abscissae are eliminated and will not propagate into the subsequent determination of astrometric parameters in the form of colour or position dependent systematic errors.

Since the sphere solution is constrained by the stellar astrometric model describing the coordinate direction in terms of the five astrometric parameters, it is important that this model actually applies to all the stars considered jointly in the sphere solution. Resolved

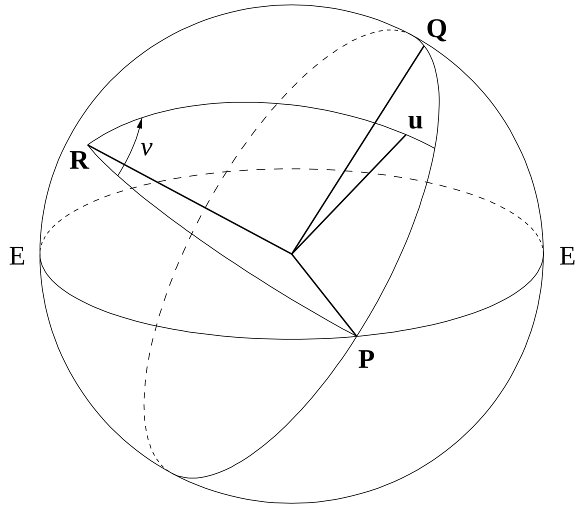


***Figure 11.1.*** *The (nominal) abscissa $v$ is defined as the angle, as seen from the pole* **R** *of the reference great circle, from the ascending node* **P** *on the fundamental plane (* EE *= equator or ecliptic) to the coordinate direction of the object,* **u**. *The vector triad* [ **P Q R** ] *defines the great-circle reference frame.*

double stars, astrometric binaries showing curved motion, and other peculiar objects, require more complex models and should therefore not be used in this process.

What is here called the sphere solution can accordingly be divided into two successive processes: (1) the sphere solution proper, which primarily aims at the accurate determination of the abscissa zero point corrections $c_j$ by means of a joint least-squares solution for a carefully selected subset of the Hipparcos stars (known as the 'primary reference stars'); and (2) the application of these corrections to all the abscissae and the subsequent determination of the astrometric parameters on a star-by-star basis—this being no longer restricted to the primary reference stars but applicable to all objects.

These processes are equivalent to the second and third steps of the so-called 'three-step method' outlined in Section 4.1. In the FAST Consortium they were executed as two separate tasks, while in NDAC they were integrated into a single task. One advantage of the FAST approach is that the second task can be made very flexible and include a variety of object models in addition to the standard five-parameter single-star model. In the NDAC Consortium all stars for which the standard model was not adopted were treated by special off-line software, sometimes completely side-stepping the three-step method, as in the case of resolved double and multiple stars (Chapter 13).

In their mathematical formulation the two steps—the sphere solution proper and the determination of astrometric parameters—are intimately connected and it is convenient to present them together. For the sake of brevity, the indices $j$ (for the reference great circles) and $i$ (for the stars) are suppressed where not explicitly needed.

## 11.2. The Reference Great-Circle Frame

The abscissae and ordinates used in the great-circle reductions are spherical coordinates, analogous to the right ascension and declination, defined with respect to a coordinate triad $\mathcal{R}$ which may be called the reference great-circle frame. Nominally the great-circle frame is uniquely defined by the celestial coordinates $(\lambda_R, \beta_R)$ of the reference great-circle pole and the fundamental celestial plane (ecliptic or equator). Formally, it may be represented by the vector triad $\mathcal{R} = [\,\mathbf{P}\ \mathbf{Q}\ \mathbf{R}\,]$, where:

$$\begin{aligned} \mathbf{R} &= \mathbf{x}\cos\beta_R\cos\lambda_R + \mathbf{y}\cos\beta_R\sin\lambda_R + \mathbf{z}\sin\beta_R \\ \mathbf{P} &= \langle \mathbf{z}\times\mathbf{R}\rangle \\ \mathbf{Q} &= \mathbf{R}\times\mathbf{P} \end{aligned} \qquad [11.1]$$

The topocentric coordinate direction of the star can be expressed in the great-circle frame as:

$$\mathbf{u} = \mathcal{R}\begin{pmatrix} \cos r\cos v \\ \cos r\sin v \\ \sin r \end{pmatrix} \qquad [11.2]$$

where $(v, r)$ are the abscissa and ordinate of the star (Figure 11.1). The topocentric coordinate direction of a star may be computed from its astrometric parameters as described in Volume 1, Section 1.2.8; given the pole of the reference great circle, the abscissa is then obtained by means of Equations 11.1 and 11.2. It is the purpose of the sphere solution to compare this calculated abscissa with the observed abscissa resulting from the great-circle reduction, in order to improve the astrometric parameters.

## 11.3. Observation Equation

The observation equation expresses the difference between the observed and calculated abscissa, $\Delta v_{ji} = v_{ji}^{\rm obs} - v_{ji}^{\rm calc}$, in terms of the different sources of error. The observation equation is in reality the same for the sphere solution proper and for the determination of the astrometric parameters; the processes differ in how the different terms are treated in the solution of the equations. Presently six kinds of error terms are considered:

- errors in the astrometric parameters;
- orientation errors in the reference great-circle frame;
- other ('local') errors on the great-circle level;
- global errors;
- grid-step errors;
- random noise.

These are discussed in subsequent subsections.

### Errors in the Astrometric Parameters

The standard model of stellar motion (Volume 1, Section 1.2.8) gives the topocentric coordinate direction at time $t$ as:

$$\mathbf{u} = \langle\, \mathbf{r}(1+\zeta t) + \mathbf{p}\mu_{\lambda*}t + \mathbf{q}\mu_\beta t - \mathbf{b}\pi/A \,\rangle \qquad [11.3]$$

where:

$$\mathbf{r} = \text{the barycentric direction of the star;}$$
$$\mathbf{p} = \langle \mathbf{z} \times \mathbf{r} \rangle = \text{the direction of } +\lambda \text{ at the star;}$$
$$\mathbf{q} = \mathbf{r} \times \mathbf{p} = \text{the direction of } +\beta \text{ at the star;}$$
$$(\mu_{\lambda*}, \mu_\beta) = \text{the components of the proper motion;}$$
$$\pi = \text{the parallax;}$$
$$\mathbf{b} = \text{the barycentric position of Hipparcos at time } t;$$
$$A = \text{the astronomical unit;}$$
$$\zeta = V_R \pi / A, \text{ where } V_R \text{ is the radial velocity of the star.}$$

[ **p q r** ] is the normal triad at **r** relative to the ecliptic coordinate system. All quantities except **b** refer to the epoch $t = 0$ (J1991.25). **b**, $A$ and $V_R$ are regarded as known; other quantities are uniquely defined by the five astrometric parameters $\lambda$, $\beta$, $\pi$, $\mu_{\lambda*}$, $\mu_\beta$, since:

$$\mathbf{r} = \mathbf{x} \cos\beta \cos\lambda + \mathbf{y} \cos\beta \sin\lambda + \mathbf{z} \sin\beta \qquad [11.4]$$

The determination of the astrometric parameters proceeds by successive differential corrections to a set of initial values. To compute the effects of small changes in the astrometric parameters it is then acceptable to ignore $\zeta$ and the normalisation brackets in Equation 11.3, yielding:

$$\Delta\mathbf{u} = \mathbf{p}(\Delta\lambda* + t\Delta\mu_{\lambda*}) + \mathbf{q}(\Delta\beta + t\Delta\mu_\beta) - \mathbf{b}A^{-1}\Delta\pi \qquad [11.5]$$

On the other hand, Equation 11.2 gives:

$$\Delta\mathbf{u} = \mathbf{m}\Delta v* + \mathbf{n}\Delta r \qquad [11.6]$$

where $\Delta v* = \Delta v \cos r$ and:

$$\mathbf{m} = \langle \mathbf{R} \times \mathbf{u} \rangle, \qquad \mathbf{n} = \mathbf{u} \times \mathbf{m} \qquad [11.7]$$

are the unit vectors in the directions of $+v$ and $+r$, respectively. [ **m n u** ] is the normal triad at **u** relative to $\mathcal{R}$. Equating $\Delta\mathbf{u}$ in Equations 11.5 and 11.6, and invoking scalar multiplication by **m** and **n**, gives:

$$\Delta v* = \mathbf{m}'\mathbf{p}(\Delta\lambda* + t\Delta\mu_{\lambda*}) + \mathbf{m}'\mathbf{q}(\Delta\beta + t\Delta\mu_\beta) - \mathbf{m}'\mathbf{b}A^{-1}\Delta\pi \qquad [11.8a]$$
$$\Delta r = \mathbf{n}'\mathbf{p}(\Delta\lambda* + t\Delta\mu_{\lambda*}) + \mathbf{n}'\mathbf{q}(\Delta\beta + t\Delta\mu_\beta) - \mathbf{n}'\mathbf{b}A^{-1}\Delta\pi \qquad [11.8b]$$

Equation 11.8b is not used. After multiplication by $\sec r$, Equation 11.8a gives the relevant terms in the observation equation, or:

$$v^{\text{obs}} - v^{\text{calc}} = \cdots + \mathbf{d}'\Delta\mathbf{a} \qquad [11.9]$$

where $\Delta\mathbf{a} = (\Delta\lambda*, \Delta\beta, \Delta\pi, \Delta\mu_{\lambda*}, \Delta\mu_\beta)'$ is the column matrix of differential corrections and **d** is the column matrix of dependencies:

$$\begin{aligned} d_1 &= \mathbf{m}'\mathbf{p} \sec r \\ d_2 &= \mathbf{m}'\mathbf{q} \sec r \\ d_3 &= \mathbf{m}'\mathbf{b}\, A^{-1} \sec r \\ d_4 &= \mathbf{m}'\mathbf{p}\, t \sec r \\ d_5 &= \mathbf{m}'\mathbf{q}\, t \sec r \end{aligned} \qquad [11.10]$$

### Orientation Errors in the Reference Great-Circle Frame

Section 11.2 defined the nominal reference great-circle frame $\mathcal{R}$, having its pole precisely at the nominal coordinates $(\lambda_R, \beta_R)$ and the abscissa origin ($\mathbf{P}$) exactly at the intersection with the ecliptic. Because of the arbitrary abscissa zero point adopted in the great-circle reduction, and because of errors in the attitude angles and stellar coordinates used as input to the great-circle reduction, the object was in reality 'observed' with respect to a slightly different triad $\tilde{\mathcal{R}} = [\, \tilde{\mathbf{P}}\ \tilde{\mathbf{Q}}\ \tilde{\mathbf{R}}\, ]$, which shall be called the actual great-circle frame. The topocentric coordinate direction of the star can be expressed in this frame as:

$$\mathbf{u} = \tilde{\mathcal{R}} \begin{pmatrix} \cos\tilde{r}\cos\tilde{v} \\ \cos\tilde{r}\sin\tilde{v} \\ \sin\tilde{r} \end{pmatrix} \qquad [11.11]$$

where $(\tilde{v}, \tilde{r})$ are the abscissa and ordinate in the nominal great-circle frame. The direction cosines in Equations 11.2 and 11.11 are related through the matrix equation:

$$\tilde{\mathcal{R}}'\mathbf{u} = (\tilde{\mathcal{R}}'\mathcal{R})\,\mathcal{R}'\mathbf{u} \qquad [11.12]$$

where $\tilde{\mathcal{R}}'\mathcal{R}$ is a $3 \times 3$ orthogonal matrix.

The relation between the nominal and actual great-circle frames can be represented by a vector $\boldsymbol{\theta}$ (unique for each great-circle reduction) such that a triad initially aligned with $\mathcal{R}$ will become aligned with $\tilde{\mathcal{R}}$ after rotation through the angle $\theta = |\boldsymbol{\theta}|$ about the unit vector $\langle\boldsymbol{\theta}\rangle$. In the small-angle approximation, neglecting terms of order $\theta^2$, this can be written:

$$\tilde{\mathcal{R}} = \mathcal{R} + \boldsymbol{\theta} \times \mathcal{R} \qquad [11.13]$$

and the transformation matrix in Equation 11.12 becomes:

$$\tilde{\mathcal{R}}'\mathcal{R} = \mathbf{I} + (\boldsymbol{\theta} \times \mathcal{R})'\mathcal{R} = \begin{pmatrix} 1 & \theta_R & -\theta_Q \\ -\theta_R & 1 & \theta_P \\ \theta_Q & -\theta_P & 1 \end{pmatrix} \qquad [11.14]$$

Here, $\mathbf{I}$ is the $3 \times 3$ identity matrix and $\theta_P$, $\theta_Q$, $\theta_R$ are the components of $\boldsymbol{\theta}$ in either great-circle frame.

Inserting Equation 11.14 in 11.12 and expanding to first order in the small angles gives:

$$\tilde{v} = v + (\theta_P \cos v + \theta_Q \sin v)\tan r - \theta_R \qquad [11.15a]$$

$$\tilde{r} = r - \theta_P \sin v + \theta_Q \cos v \qquad [11.15b]$$

At this point two simplifications are introduced:

(1) since the ordinate was not estimated in the great-circle reduction, Equation 11.15b need not be considered;

(2) since $|r| \lesssim 2$ degrees, due to the limited time interval of the great-circle reduction and the choice of the reference great circle close to the mean scanning plane during that interval, the components $\theta_P$ and $\theta_Q$ contribute much less than $\theta_R$ to the difference between the nominal and actual abscissa in Equation 11.15a, and are ignored.

(1) implies a small loss of information, but involves no approximation compared with Equation 11.15; in contrast, (2) causes an approximation error in the abscissae which could amount to a few milliarcsec (since $\theta_P$ and $\theta_Q$ may be of the order of the accuracy of the transverse attitude, or 0.1 arcsec). It was assumed that the outer iteration loop

of the main Hipparcos reductions—involving the attitude determination, great-circle reductions, and the sphere solution—eliminates at least the systematic part of these errors. The consequences of this approximation are further discussed in Section 11.7.

As a result of (1) and (2), Equation 11.15 simplifies to $\tilde{v} = v - \theta_R$ and $\theta_R$ can be identified with the zero-point correction $c_j$ that must be added to the observed abscissa (in the actual great-circle frame) in order to be compared with the calculated abscissa (in the nominal frame). The corresponding term in the observation equation is, therefore:

$$v^{\mathrm{obs}} - v^{\mathrm{calc}} = \cdots - c_j \qquad [11.16]$$

### Local Errors on the Great-Circle Level

Apart from the orientation errors of the great-circle frame, the abscissae may be subject to various distortions and systematic displacements, which vary from one great-circle reduction to the next. This kind of 'local' error was not originally foreseen in the Hipparcos data reductions, and are therefore not described in the pre-launch documentation (Perryman *et al.* 1989 Volume III). Experiments with the real data, in particular FAST/NDAC comparisons made at the great-circle level and the analysis of residuals from several provisional sphere solutions, clearly demonstrated that such effects existed. The most important one seemed to be a periodic error in the abscissa, with a period of 60°, and with essentially random amplitudes and phases in the different great-circle reductions. The source of this could simply be the relatively low rigidity of the great-circle reductions to the sixth harmonic of the abscissae, due to the proximity of the basic angle (58°) to the period of that harmonic. The 'local' sixth harmonic may be introduced into the observation equation in the form of the following two terms:

$$v^{\mathrm{obs}} - v^{\mathrm{calc}} = \cdots + C_j \cos 6(v - v_\odot) + S_j \sin 6(v - v_\odot) \qquad [11.17]$$

where $v_\odot$ is the abscissa of the Sun, which for historical reasons was taken as the origin for the phase of the harmonic errors.

Additional local errors, especially depending on colour, were also detected and taken into account in some of the sphere solutions (see Section 16.3).

### Global Errors

Global parameters $\Gamma_k$, $k = 1 \ldots N_\Gamma$ were primarily introduced in order to take into account instrumental effects which could not be resolved at the level of the great-circle reductions. In the various sphere solutions they varied in kind and number, up to $N_\Gamma \simeq 20$, as the physical significance and mathematical form of the effects were explored.

By far the most important instrumental effect requiring global treatment was the so-called 'constant chromaticity'. In the Hipparcos nomenclature, this was the average value of the displacement of the image of a star of given colour index with respect to the image of a star of colour $B - V = 0.5$ mag. The displacement was measured in the direction of scanning, and the average taken over both fields of view. Assuming that the displacement was proportional to the difference in colour index, the relevant term in the observation equation was:

$$v^{\mathrm{obs}} - v^{\mathrm{calc}} = \cdots + (B - V - 0.5)\Gamma_{\mathrm{chrom}} \qquad [11.18]$$

It was however found that the chromaticity varied (linearly) with time, requiring one more global parameter for its representation, and that the variation with colour index was perhaps not linear, requiring yet another parameter. The actual parameters used by FAST and NDAC in their successive sphere solutions are described in Chapter 16.

Another kind of global instrumental effect was foreseen as a consequence of the varying thermal impact on the payload. Under the nominal scanning law the solar illumination varied periodically with the spin phase relative to the Sun, i.e. the heliotropic angle $\Omega$ (see Figure 7.3). Consequently it was assumed that systematic thermal variations could be modelled as a periodic function in $\Omega$. Systematic errors in the abscissae caused by such variations must be periodic in $v - v_\odot$, if $v_\odot$ is the abscissa of the Sun. This reasoning lead to the introduction of global parameters with harmonic coefficients $\cos n(v - v_\odot)$ $(n = 1 \ldots 6)$ and $\sin n(v - v_\odot)$ $(n = 2 \ldots 6)$. The term containing $\sin(v - v_\odot)$ was rejected *a priori*, as it would have a very strong correlation with the parallax zero point. Subsequently it was found that none of these global harmonic parameters attained significant amplitudes. They were abandoned in the later NDAC solutions; in the final FAST solution their amplitudes were below 0.01 mas (Table 16.3). The sixth harmonic was however found to be an important local error, i.e. with independent coefficients for each great circle, as discussed in the previous subsection.

Some sphere solutions included a global parameter representing a correction to the general-relativistic value of the gravitational light deflection in the heliocentric metric. This parameter was introduced because Hipparcos offered the first opportunity to measure the deflection accurately, for optical wavelengths, at large angles from the Sun. According to General Relativity, for an object at infinity, the projection of the deflection onto the reference great circle, or the difference in abscissa between the natural direction and the coordinate direction to the star, is given by:

$$\Delta v_{\mathrm{GR}} = \frac{2GS}{h_0 c^2} \frac{\mathbf{u}'_\odot \langle \mathbf{u} \times \mathbf{R} \rangle}{1 - \mathbf{u}'_\odot \mathbf{u}} \qquad [11.19]$$

where $GS$ is the heliocentric gravitational constant (Table 12.1), $h_0$ the distance from the Sun to the observer, and $\mathbf{u}_\odot$ the coordinate direction towards the Sun; the latter two are computed from the heliocentric position of the observer, $\mathbf{h}_0 = \mathbf{b}_0 - \mathbf{b}_{\mathrm{S}}$, as $h_0 = |\mathbf{h}_0|$ and $\mathbf{u}_\odot = -\langle \mathbf{h}_0 \rangle$. The global parameter may be defined in terms of the PPN parameter $\gamma$ as $\Gamma_{\mathrm{GR}} = \gamma - 1$, in which case the relevant coefficient in the observation equation is $\Delta v_{\mathrm{GR}}/2$. A slightly different definition was used by NDAC (see Equation 16.10).

Irrespective of the choice and precise definition of global parameters, the corresponding terms in the observation equation can be expressed as:

$$v^{\mathrm{obs}} - v^{\mathrm{calc}} = \cdots + \mathbf{g}'\boldsymbol{\Gamma} \qquad [11.20]$$

where $\boldsymbol{\Gamma} = (\Gamma_1, \ldots, \Gamma_{N_\Gamma})'$ is the column matrix of global parameters and $\mathbf{g}$ is the column matrix of coefficients.

### Grid-Step Errors

The abscissa resulting from the great-circle reduction was sometimes wrong by a small multiple of the grid step, due to the 360° phase ambiguity of the signal produced by the modulating grid. In the observation equation the presence of grid-step errors is accounted for by the term:

$$v^{\mathrm{obs}} - v^{\mathrm{calc}} = \cdots + ns \qquad [11.21]$$

where $n$ is a small integer (usually $n = 0$) and $s = 1.2074$ arcsec is the adopted mean value of the grid step.

### Random Noise

The observation equation is completed by adding a noise term $\eta$ representing the random part of the observational errors resulting from the great-circle reductions. This was assumed to be centred (expected value $E(\eta) = 0$), essentially Gaussian (although outliers were expected and had to be accommodated by the solution method), and of a standard deviation $\sigma_v$ which was basically known from the great-circle reduction. Furthermore, the noise was assumed to be uncorrelated. This is known to be false, in general, for a pair of abscissae obtained in the same great-circle reduction (see Figures 16.36–16.37), but it is a reasonable assumption for the different abscissae of a given star obtained in different orbits.

The abscissa standard errors, $\sigma_v$, were estimated as part of the great-circle reductions. However, it was empirically found that these estimates in general required corrections, either in the form of a multiplicative factor, an added variance, or a combination of both; and which were often found to be functions of magnitude, colour and time. Such corrections were derived from the unit-weight variance of the residuals of the sphere solution, first considered individually by the data reduction consortia (see Sections 11.5 and 11.6), and finally as part of the merging of the consortia results (Chapter 17).

### Complete Observation Equation

The abscissa zero point correction $c_j$ and the other (local) errors on the great-circle level (e.g. $C_j$, $S_j$) may be brought together in a single unknown column matrix $\mathbf{c}_j$ for each great-circle reduction, containing $1 \le n_c \lesssim 3$ elements. The corresponding coefficient matrix, also of length $n_c$, is denoted $\mathbf{e}_{ji}$. In the simplest case of $n_c = 1$, the only element in $\mathbf{c}_j$ is $c_j$, and the coefficient matrix is $\mathbf{e}_{ji} = (-1)$.

Combining the error terms gives the complete observation equation:

$$\mathbf{d}'_{ji}\Delta\mathbf{a}_i + \mathbf{e}'_{ji}\mathbf{c}_j + \mathbf{g}'_{ji}\mathbf{\Gamma} + n_{ji}s + \eta_{ji} = v_{ji}^{\mathrm{obs}} - v_{ji}^{\mathrm{calc}} \qquad [11.22]$$

where the calculated (nominal) abscissa, obtained through Equations 11.2 and 11.3, is uniquely a function of the time associated with the observation ($t_{ji}$), the nominal pole of the reference great circle ($\mathbf{R}_j$), and the assumed astrometric parameters of the star ($\mathbf{a}_i$).

## 11.4. The Sphere Solution Proper

### Primary Reference Stars

The sphere solution proper aims at a direct solution of the system of observation equations, Equation 11.22, the main objective being the estimation of the abscissa zero points ($c_j$) and the global parameters ($\mathbf{\Gamma}$). As already explained, this objective was achieved using only a subset of all the observation equations, corresponding to the 'primary reference stars'. The selection of primary reference stars was guided by the following considerations.

In the right-hand side of Equation 11.22, all terms are less than a few arcseconds, or $\simeq 10^{-5}$ rad. Linearisation errors were therefore of the order of $10^{-10}$ rad $\simeq 0.02$ mas, and could be neglected. However, the presence of the grid-step term $n_{ji}s$ still made the system of observation equations highly non-linear and unsuitable for direct solution by standard (least-squares) methods. It was therefore necessary to restrict the sphere solution proper to objects with good *a priori* positions, for which $n_{ji} = 0$ could be assumed with a high degree of confidence.

The standard model of stellar motion, Equation 11.3, is only valid for stars which, from the viewpoint of the Hipparcos observations, could be regarded as point objects with uniform motion. This excludes well-resolved binaries and multiple stars, for which the abscissa derived from the phases of the detector signal is a complicated function of the geometry of the system, the relative intensity of the components, and the direction of scanning. It also excludes close binaries, where the photocentre shows a non-negligible acceleration due to the orbital motion of the system. Known double and multiple stars of such characteristics were therefore excluded *a priori*.

The choice of primary reference stars for the FAST sphere solutions was essentially made *a priori* according to these criteria. It was also attempted to use only photometrically constant stars with good coverage. Within these restrictions it was, furthermore, desirable to have an even distribution over the whole celestial sphere, preferably with at least one primary reference star per square degree. This lead to the use of approximately 72 000 primary reference stars in the final iterations of the sphere solutions. In NDAC, a first choice was made according to the above considerations of duplicity and possible grid-step errors, and further stars were rejected while setting up the observation equations, on the basis of the residuals with respect to the previous iteration of the sphere solution. This resulted in some 50 000 primary stars in the early sphere solutions, increasing to about 78 000 in the final sphere solution.

### General Problem

Due to the selection of primary reference stars, the grid-step term can be disregarded for the sphere solution proper. The remaining unknowns fall into three groups depending on their different scope of validity:

for each primary reference star ($i$): $\Delta\mathbf{a}_i$

for each great-circle frame ($j$): $\mathbf{c}_j$

for each observation ($ji$): $\mathbf{\Gamma}$

The structure of the observation equations, and hence the methods of solution, are strongly influenced by this categorisation.

Before solving the equations, it was necessary to equalise their statistical weights. This was done by dividing each equation by $\sigma_{v_{ji}}$, the actual standard error of the observation (empirically corrected as described in Section 11.6). The resulting equations can be written in matrix form as:

$$\mathbf{A}\,\mathbf{\Delta a} + \mathbf{C}\,\mathbf{c} + \mathbf{G}\,\mathbf{\Gamma} + \boldsymbol{\eta} = \mathbf{\Delta v} \qquad [11.23]$$

where $\mathbf{\Delta a}$, $\mathbf{c}$ and $\mathbf{\Gamma}$ are column matrices with the three kinds of unknowns; they are of length $5N_p$, $n_c N_c$ and $N_\Gamma$, respectively. $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{G}$ are the corresponding design matrices, obtained from the submatrices $\mathbf{d}'_{ji}$, $\mathbf{e}'_{ji}$ and $\mathbf{g}'_{ji}$ in Equation 11.22 after division by $\sigma_{v_{ji}}$. $\mathbf{\Delta v}$ is the column matrix of abscissa differences (observed minus calculated, and normalised to unit weight), and $\boldsymbol{\eta}$ is a column matrix of noise with assumed covariance $\mathrm{E}(\boldsymbol{\eta}\boldsymbol{\eta}') = \mathbf{I}$. The number of rows in $\mathbf{A}$, $\mathbf{C}$, $\mathbf{G}$, $\boldsymbol{\eta}$ and $\mathbf{\Delta v}$ is equal to $M_p$, the number of observations (abscissae) for the primary reference stars.

The general problem of the sphere solution proper was to find the vectors $\mathbf{\Delta a}$, $\mathbf{c}$ and $\mathbf{\Gamma}$ which minimised the Euclidean (L2) norm of the residuals, or:

$$\min \; \| \mathbf{A}\,\mathbf{\Delta a} + \mathbf{C}\,\mathbf{c} + \mathbf{G}\,\mathbf{\Gamma} - \mathbf{\Delta v} \|_2 \qquad [11.24]$$

The size of the problem can be appreciated by considering the number of unknowns and equations in the final sphere solutions (Table 11.1). The matrix ( $\mathbf{A}\ \mathbf{C}\ \mathbf{G}$ ), known as the design matrix of the least-squares problem, was however very sparse: in each row, only five elements in $\mathbf{A}$, one to three elements in $\mathbf{C}$ and $N_\Gamma$ elements in $\mathbf{G}$ were, by design, different from zero. The filling factor was, therefore, $(5+n_c+N_\Gamma)/M_p \simeq 8\times 10^{-6}$ for solution F37.3 and $\simeq 4\times 10^{-6}$ for solution N37.5 (see Table 16.1 for details of the sphere solution nomenclature). The structure of the design matrix is illustrated in Figure 11.2.

A feature of the sphere solution problem is that the reference frame for the astrometric parameters and the abscissa zero points remains unspecified by the observations. This should in principle result in a six-fold singularity of the system of observation equations, corresponding to the six degrees of freedom of the reference frame. In reality it was found that Equation 11.23 was not singular; this problem is further discussed in Section 11.7. Nevertheless, in the practical implementation of the sphere solution it was necessary to consider the theoretical rank deficiency especially for the calculation of the variances.

### Implementation in FAST

Two basic algorithms were developed in FAST to perform the sphere solution. Before the launch of the satellite, a working solution was tested and fully implemented into an operational software by a team of the University of Bologna (Galligani *et al.* 1989). This software used the iterative algorithm LSQR based on the Lanczos method, which was chosen after various trials and adapted to solve the large-scale system of the sphere solution. It met at that time the stringent requirements set by the computer resources in the mid-eighties. While it gave satisfactory solutions, it had two major drawbacks:

1. it was to be used as a ‘black box’ algorithm and lacked the necessary flexibility required during the processing of real data to cope with new modelling, the need to make an *a priori* selection of observations, and to produce various statistics;
2. a reliable estimate of the covariance matrix was very difficult to achieve and depended on the iteration scheme adopted.

To overcome these shortcomings, in particular in view of getting a good estimate of the internal precision of the solution parameters, a second method was developed at CERGA (Frœschlé 1992). This method, based on a block iteration scheme, proved to be very flexible and was easily adapted to a changing environment, as the knowledge of the true properties of the data became more refined with time. The LSQR software was run extensively in parallel during the development phase of this alternative method and helped to speed up the tuning of the new software.

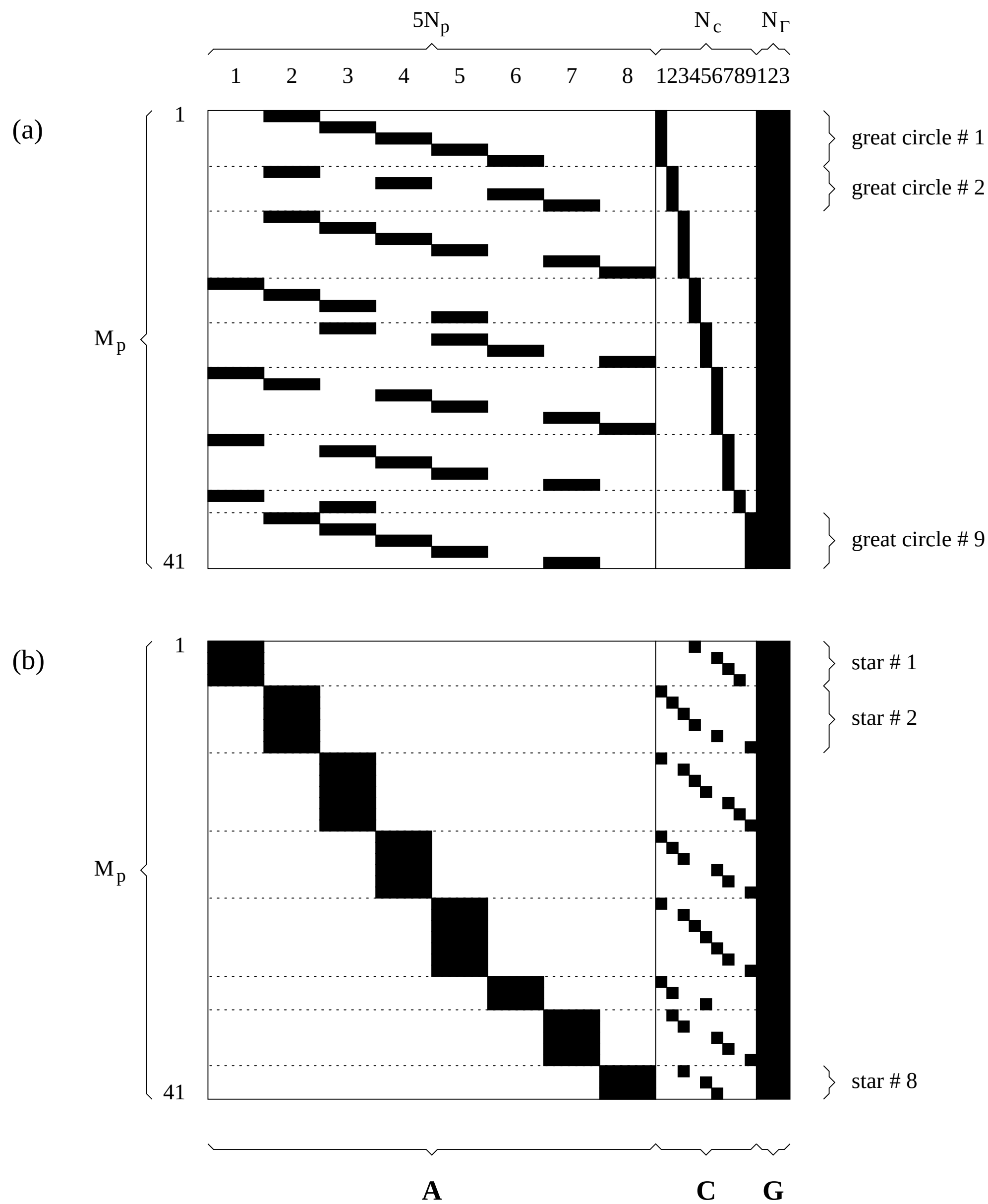


***Figure 11.2.*** *Schematic illustration of the structure of the design matrix* ( **A C G** ) *for a case with* $N_p = 8$ *primary reference stars* ($5N_p = 40$ *astrometric parameters*), $N_c = 9$ *reference great circles (each with* $n_c = 1$ *unknown, namely the abscissa zero point*), $N_\Gamma = 3$ *global parameters, and* $M_p = 41$ *observations (abscissae) referring to the primary reference stars. The black areas are the non-zero elements of the design matrix. In the upper diagram (a) the observations are ordered by the great-circle number (i.e. more-or-less chronologically); in the lower diagram (b) by the star number. Actual numbers* $M_p$, $N_p$, $N_c$ *and* $N_\Gamma$ *are given in Table 11.1.*

**Table 11.1.** Number of equations and unknowns in the final sphere solutions F37.3 (FAST) and N37.5 (NDAC). Only data corresponding to the primary reference stars are considered. See Chapter 16 for further details on these solutions.

| Solution | F37.3 | N37.5 |
|---|---|---|
| Number of equations, $M_p$ | 2 091 926 | 2 451 483 |
| Number of unknowns: | | |
| astrometric parameters, $5N_p$ | 362 455 | 390 565 |
| great-circle zero points, $N_c$ | 2 281 | 2 326 |
| other local parameters, $(n_c - 1)N_c$ | 4 562 | – |
| global parameters, $N_\Gamma$ | 8 | 3 |
| total number, $5N_p + n_c N_c + N_\Gamma$ | 369 306 | 392 894 |

In linearised form the observation equations are written:

$$\mathbf{C}\,\delta\mathbf{c} + \mathbf{A}\,\delta\mathbf{a} + \mathbf{G}\,\delta\boldsymbol{\Gamma} + \boldsymbol{\eta} = \delta\mathbf{v} \qquad [11.25]$$

where $\delta\mathbf{c}$, $\delta\mathbf{a}$, $\delta\boldsymbol{\Gamma}$ are differential corrections to the local circle parameters, the astrometric parameters, and the global parameters, respectively. This is the order in which the unknowns are solved by the block iteration scheme; hence the exchange of the first two terms compared to Equation 11.23. The harmonic coefficients $C_j$ and $S_j$ (Equation 11.17) were included among the circle parameters, so the lengths of the correction vectors were $3N_c$, $5N_p$ and $N_\Gamma$.

The block iteration scheme operated on (parts of) the normal equations obtained by the least-squares method:

$$\begin{pmatrix} \mathbf{C}'\mathbf{C} & \mathbf{C}'\mathbf{A} & \mathbf{C}'\mathbf{G} \\ \mathbf{A}'\mathbf{C} & \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{G} \\ \mathbf{G}'\mathbf{C} & \mathbf{G}'\mathbf{A} & \mathbf{G}'\mathbf{G} \end{pmatrix} \begin{pmatrix} \delta\mathbf{c} \\ \delta\mathbf{a} \\ \delta\boldsymbol{\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{C}'\delta\mathbf{v} \\ \mathbf{A}'\delta\mathbf{v} \\ \mathbf{G}'\delta\mathbf{v} \end{pmatrix} \qquad [11.26]$$

where the dimension of the normal matrix is $N \times N$ with $N = 3N_c + 5N_p + N_\Gamma \simeq 370\,000$. No direct and general method of resolution could be reasonably envisioned for a system of this size. The way out was to take advantage of the block structure of $\mathbf{A}$ and of the fact that $\mathbf{C}$ is a sparse matrix. If there were only the astrometric unknowns the problem would reduce to solving as many $5 \times 5$ linear systems as there are stars, which is an easy task. The block decomposition attempts to solve more-or-less independently the unknowns related to the stars and those linked to the more general parameters. This leads to a very natural iterative design, but has the drawback of disregarding the cross-correlations between the astrometry and the general parameters.

In a first approximation one considers that the corrections $\delta\mathbf{a}$ and $\delta\boldsymbol{\Gamma}$ are negligibly small. The harmonic terms $C_j = \mathbf{c}_{j2}$ and $S_j = \mathbf{c}_{j3}$ are also neglected in this approximation. Then the matrix $\mathbf{C}$ is sorted according to the great-circle index and the correction to the abscissa origin $\delta\mathbf{c}_{j1}$ is simply the average of the $\delta v_{ji}$ for that reference great circle. Denote by $\mathcal{I}_j$ the set of primary reference stars observed with respect to great circle $j$, and let $N_j$ be the number of such stars. The zero order solution is then given by:

$$\begin{aligned} &\delta\mathbf{c}^{(0)}_{j1} = N_j^{-1} \sum_{i\in\mathcal{I}_j} \delta v_{ji}, \quad \delta\mathbf{c}^{(0)}_{j2} = 0, \quad \delta\mathbf{c}^{(0)}_{j3} = 0, \quad j = 1 \ldots N_c \\ &\delta\mathbf{a}^{(0)} = \mathbf{0} \\ &\delta\boldsymbol{\Gamma}^{(0)} = \mathbf{0} \end{aligned} \qquad [11.27]$$

where the sum is taken over the stars $i$ included in great-circle reduction $j$. The corrections to the astrometric parameters are then computed, star by star, resulting in the approximation:

$$\begin{aligned} \delta\mathbf{c}^{(1)} &= \delta\mathbf{c}^{(0)} \\ \delta\mathbf{a}_i^{(1)} &= (\mathbf{A}_i'\mathbf{A}_i)^{-1}\mathbf{A}_i' \left[\delta\mathbf{v}_i - \mathbf{C}_i\delta\mathbf{c}^{(0)}\right], \quad i = 1\ldots N_p \\ \delta\mathbf{\Gamma}^{(1)} &= \mathbf{0} \end{aligned} \qquad [11.28]$$

where $\mathbf{A}_i$ and $\mathbf{C}_i$ are the blocks of $\mathbf{A}$ and $\mathbf{C}$ associated with the star $i$, and $\delta\mathbf{v}_i$ is the corresponding observations. Then:

$$\begin{aligned} \delta\mathbf{c}^{(2)} &= \delta\mathbf{c}^{(1)} \\ \delta\mathbf{a}^{(2)} &= \delta\mathbf{a}^{(1)} \\ \delta\mathbf{\Gamma}^{(2)} &= (\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}' \left[\delta\mathbf{v} - \mathbf{C}\,\delta\mathbf{c}^{(1)} - \mathbf{A}\,\delta\mathbf{a}^{(1)}\right] \end{aligned} \qquad [11.29]$$

Equations 11.27–11.29 were iterated until convergence. The other local parameters ($C_j$ and $S_j$) were introduced from the second iteration. There were two stopping criteria tested at every step: (1) a normalised $\chi^2$ based on the residuals left at every observation, and (2) the variation from one iteration to the next of the corrections to the origins.

The sphere solution in FAST was kept completely free to rotate and no attempt was made to remove the rank deficiency (see Section 11.7 for a discussion). Various experiments were made at intermediate stages to constrain the system by fixing the position and proper motion of '$1\frac{1}{2}$ star', e.g. the longitude and latitude of one star and the latitude of a second, thus removing the theoretical six degrees of freedom. But the linear system of the sphere solution was in fact not singular but only mildly ill-conditioned and the constraints brought no decisive advantage. In addition the variance-covariance matrix of the astrometric parameters was to be recomputed later with an independent software and it was not a major concern during the sphere solution proper to obtain realistic variances.

### Implementation in NDAC

The solution to the general problem of Equation 11.24 was implemented in NDAC by way of the normal equations. Only one local parameter was used for each great circle ($n_c = 1$), so the complete normal equations matrix system had $5N_p + N_c + N_\Gamma \simeq 400\,000$ unknowns. This was reduced to a manageable size of $N_c + N_\Gamma \simeq 2300$ by eliminating the astrometric parameters in parallel with the accumulation of the normal equations for the remaining parameters. This required that the observation equations were ordered according to the star numbers as in Figure 11.2b. Since the abscissae were received from the great-circle reductions in the order in which those reductions had been made, a first part of the sphere solution consisted of the sorting of all the abscissa data according to the star numbers.

For the subsequent formulation there is no need to distinguish between the abscissa zero points and the global parameters, as they were treated together as a single column matrix $\mathbf{b}$ with $N_b = N_c + N_\Gamma$ rows. In order to eliminate outliers the calculation of all the unknowns was actually made by a sequence of differential corrections $\delta\mathbf{a}$, $\delta\mathbf{b}$ to the initial values. Introducing the matrix $\mathbf{B} = (\,\mathbf{C}\ \mathbf{G}\,)$ of dimension $M_p \times N_b$ the observation equations for the corrections are written:

$$\mathbf{A}\,\delta\mathbf{a} + \mathbf{B}\,\delta\mathbf{b} + \boldsymbol{\eta} = \delta\mathbf{v} \qquad [11.30]$$

and the full system of normal equations is:

$$\begin{pmatrix} \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{B} \\ \mathbf{B}'\mathbf{A} & \mathbf{B}'\mathbf{B} \end{pmatrix} \begin{pmatrix} \delta\mathbf{a} \\ \delta\mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{A}'\delta\mathbf{v} \\ \mathbf{B}'\delta\mathbf{v} \end{pmatrix} \qquad [11.31]$$

Elimination of the stellar unknowns, $\delta\mathbf{a}$, gives the following two systems:

$$\left[\mathbf{B}'\mathbf{B} - \mathbf{B}'\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}\right]\delta\mathbf{b} = \mathbf{B}'\delta\mathbf{v} - \mathbf{B}'\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\delta\mathbf{v} \qquad [11.32a]$$

$$(\mathbf{A}'\mathbf{A})\delta\mathbf{a} = \mathbf{A}'\delta\mathbf{v} - \mathbf{A}'\mathbf{B}\delta\mathbf{b} \qquad [11.32b]$$

The $5N_p \times 5N_p$ matrix $\mathbf{A}'\mathbf{A}$ is block-diagonal, i.e. zero everywhere except for the $N_p$ blocks of size $5 \times 5$ along the diagonal. It is therefore a straightforward process to compute the two vectors:

$$\widetilde{\delta\mathbf{a}} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\delta\mathbf{v} \qquad [11.33]$$

and:

$$\widetilde{\delta\mathbf{v}} = \delta\mathbf{v} - \mathbf{A}'\widetilde{\delta\mathbf{a}} \qquad [11.34]$$

whereupon Equation 11.32a can be written as:

$$\left[\mathbf{B}'\mathbf{B} - \mathbf{B}'\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}\right]\delta\mathbf{b} = \mathbf{B}'\widetilde{\delta\mathbf{v}} \qquad [11.35]$$

The symmetric matrix on the left-hand side is of size $N_b \times N_b$ and practically filled, since almost any pair of reference great circles shared at least one primary reference star.

Once the observations had been ordered according to the star numbers, Equation 11.33 was used to compute provisional corrections to the astrometric parameters, after which Equation 11.34 gave the corresponding provisional abscissa residuals. This was done for one star at a time, while sequentially reading the sorted data into computer memory. Concurrently with this process, Equation 11.35 was accumulated. This system was complete when all the stars had been processed, and $\delta\mathbf{b}$ could then be solved by means of the Cholesky algorithm. After updating of the abscissa zero points and global parameters, the process started again with new provisional corrections to the astrometric parameters. This iteration ended when the correction vector $\delta\mathbf{b}$ was negligible: typically the updates to $c_j$ were then less than $10^{-3}$ mas. At that time the astrometric parameters had also reached their final values, as can be seen by comparing Equations 11.32b and 11.33.

It should be noted that the above process is not an iterative solution of the normal equations (Equation 11.31) but a direct solution through rigorous elimination of $\delta\mathbf{a}$. The iteration scheme was primarily needed to handle outliers among the abscissa data. The 'pre-adjustment' of the astrometric parameters, by means of the provisional updates $\widetilde{\delta\mathbf{a}}$, had some additional advantages:

- pre-adjustment was not restricted to the primary reference stars, but was in fact made for as many stars as possible, thus eliminating the need for a separate process for the determination of the astrometric parameters;
- the final decision whether to accept a star as a primary reference star could be made immediately after the pre-adjustment, partly based on an examination of the (provisional) residuals $\widetilde{\delta\mathbf{v}}$. In practice only stars with very clean residuals were accepted as primary reference stars, and the corresponding data were then added to the normals for $\delta\mathbf{b}$;
- for non-primary reference stars, the pre-adjustment stage was a convenient place to detect and correct grid-step errors, as described in Section 11.6.

The abscissa residuals after convergence were statistically analysed in a number of ways, in particular as functions of colour, magnitude, and the abscissa difference with respect to the Sun, $v - v_\odot$. This revealed a number of systematic patterns, in particular the sixth harmonic in $v - v_\odot$, with apparently independent and random coefficients (of a few milliarcsec) in the different great-circle reductions, and the chromatic effects discussed in Section 16.3. These effects were treated in an *ad hoc* manner. For the sixth harmonic and the chromatic variation, the relevant coefficients were determined from the residuals of the penultimate solution (N37.4, see Section 16.3) and subtracted from the right-hand sides of the observation equations of the final sphere solution. In a sense this resembles the block iteration scheme adopted by FAST, but it was only used for those parameters that were not included in the formal observation equations.

The system of normal equations for $\delta\mathbf{b}$ was found to have a condition number $\kappa \simeq 2300$. Thus it could be solved without adding any constraint (such as fixing the position and proper motion for '$1\frac{1}{2}$ star') with a moderate loss of numerical precision. In fact, most of this loss corresponded to the random selection of one particular solution from the manifold of solutions consistent with the observation equations, and was not accompanied by a corresponding deterioration of the reference frame. It did however result in large formal variances for the abscissa zero points and strong correlations between them, artifacts of the (almost) undefined state of rotation with respect to an external coordinate system. This problem was eliminated by projecting the solution onto the subspace which is complementary to the theoretical null space, and transforming the covariance matrix accordingly. This is practically equivalent to a minimum-norm solution and to using the pseudo-inverse for the covariances.

The minimum-norm solution was implemented as part of the Cholesky algorithm for the solution of Equation 11.35. Let $\mathbf{F}$ be the (upper-diagonal) Cholesky factor of the normal equations matrix, so that the direct solution is $\delta\mathbf{b} = \mathbf{F}^{-1}(\mathbf{F}^{-1})'\mathbf{B}'\widetilde{\delta\mathbf{v}}$ with formal covariance $\mathbf{V} = \mathbf{F}^{-1}(\mathbf{F}^{-1})'$. Furthermore let $\mathbf{N}$ be an $N_b \times 6$ matrix containing, in the six columns, a set of vectors spanning the theoretical null space. According to Equation 11.54 these are most easily constructed by taking, as the elements in row $j$, the six components of $\mathbf{R}_j$ and $\mathbf{R}_j t_j$, where $j$ is the great-circle number. A set of orthonormal vectors $\hat{\mathbf{N}}$ can be computed e.g. by the Modified Gram-Schmidt algorithm. The minimum-norm solution is then obtained by the transformation:

$$\delta\hat{\mathbf{b}} = \delta\mathbf{b} - \hat{\mathbf{N}}'\delta\mathbf{b} \qquad [11.36]$$

and the covariance of the transformed vector is:

$$\hat{\mathbf{V}} = (\mathbf{I} - \hat{\mathbf{N}}')\mathbf{V}(\mathbf{I} - \hat{\mathbf{N}}) = \left[\mathbf{F}^{-1} - \hat{\mathbf{N}}'\mathbf{F}^{-1}\right]\left[\mathbf{F}^{-1} - \hat{\mathbf{N}}'\mathbf{F}^{-1}\right]' \qquad [11.37]$$

It is seen that the inverted Cholesky factor must simply be transformed exactly like the solution vector, before the covariance matrix is formed. In practice only the diagonal elements of $\hat{\mathbf{V}}$ were computed. The standard errors of the abscissa zero points were typically about 0.1 mas.

## 11.5. Determination of Astrometric Parameters in NDAC

### General Problem

The sphere solution proper determined the abscissa zero points $c_j$ and global parameters $\mathbf{\Gamma}$ by elimination of the astrometric parameters from the basic observation equation (Equation 11.22). Shifting to the right-hand side the terms thus determined gives:

$$\mathbf{d}'_{ji}\Delta\mathbf{a}_i + n_{ji}s + \eta_{ji} = v^{\rm obs}_{ji} - v^{\rm calc}_{ji} - \mathbf{e}'_{ji}\mathbf{c}_j - \mathbf{g}'_{ji}\mathbf{\Gamma} \qquad [11.38]$$

In contrast to the original system, this can be solved directly for one star at a time, requiring only a very small system of equations to be handled at a time. However, there are still many complications to be considered, in particular the possible grid-step errors ($n_{ji} \neq 0$), deviations from the standard astrometric model (Equation 11.3) for some stars, and the existence of outliers caused, for example, by the superposition of chance stars in the instantaneous field of view from the other viewing direction.

### Implementation in NDAC

In NDAC the determination of the astrometric parameters was integrated with the sphere solution, as described in the previous section, for all stars except those treated by the special double-star process described in Chapter 13. Other cases where the standard stellar model was not applicable, principally the astrometric binaries requiring quadratic, cubic or orbital solutions for the motion of the photocentre, were not systematically investigated but the NDAC data were used for such solutions as part of the merging process (Chapter 17).

An attempt to eliminate grid-step errors was made as soon as more than one observation of the star had been rejected, or if the goodness-of-fit for the star exceeded a given threshold. As a first attempt, the integers $n_{ji}$ were chosen in such a way that:

$$|v^{\rm obs}_{ji} - v^{\rm calc}_{ji}| \leq s/2 \qquad [11.39]$$

for all the observations of this star. If the residuals were still not acceptable, a systematic search was made to determine the correct set of integers $n_{ji}$. The initial coordinates of the star were modified in steps of about 0.5 arcsec, new integers determined according to Equation 11.39, and the residuals and goodness-of-fit were again computed. This process was repeated until a satisfactory fit was obtained, or until the modified coordinates were too far away from the initial position. Usually the search was limited to an area of only a few arcsec radius, because of the high risk of finding spurious fits at larger distances.

The complete expression for the variance-covariance matrix associated with the astrometric parameters of star number $i$, obtained from Equation 11.32b, is:

$$\mathbf{V}_i = (\mathbf{A}'_i\mathbf{A}_i)^{-1} + \mathbf{S}_i\mathbf{S}'_i \qquad [11.40]$$

where $\mathbf{A}_i$ is the submatrix of $\mathbf{A}$ referring to the star, and:

$$\mathbf{S}_i = (\mathbf{A}'_i\mathbf{A}_i)^{-1}\mathbf{A}_i\mathbf{B}(\mathbf{F}^{-1} - \hat{\mathbf{N}}\mathbf{F}^{-1}) \qquad [11.41]$$

The term $\mathbf{S}_i \mathbf{S}_i'$ was, for practical reasons, neglected. This is tantamount to neglecting the variance contributed by the abscissa zero points through the second term in Equation 11.32b. This was believed to be an acceptable approximation in view of the rather small (~ 0.1 mas) errors on the abscissa zero points, compared to the typical abscissa standard errors (~ 3 mas).

## 11.6. Determination of Astrometric Parameters in FAST

The sphere solution in FAST was intended to produce the parameters required to define the system, in such a way that every abscissa could be brought into a fully consistent reference frame. The only remaining degrees of freedom were the six parameters needed for the time dependent rotation, to be determined by the link to the extragalactic reference frame. The astrometric parameters resulting from the sphere solution were considered as a by-product of this process and not as final for these stars. In any case, an independent software had to be written for the determination of the astrometric parameters of the non-primary reference stars, which were not part of the sphere solution. This software needed to be more flexible than the corresponding one in the sphere solution in order to handle all the difficult cases, the double and multiple stars, and to cope with the grid-step errors very common for stars with poor initial positions or proper motions. This led, at an early stage of the definition of the FAST organisation, to the identification of the astrometric parameter determination as a task by itself, independent of the sphere solution and to be designed to produce the astrometric solutions for all the stars.

### Environment and Main Goals

The sphere solution in the FAST processing ended up with a file containing the corrections to be applied to each origin, one per circle, so that the resulting network of circles determined a consistent reference frame on the sphere. Then all the abscissae, of the primary reference stars as well as all the other stars and solar system objects, were referred to the new origins and corrected for the general parameters. The corrected abscissae for star $i$ were:

$$\delta\tilde{\mathbf{v}}_i = \delta\mathbf{v}_i - \mathbf{C}_i\,\delta\mathbf{c} - \mathbf{G}\,\delta\mathbf{\Gamma} \qquad [11.42]$$

In the normal case of a single star following the standard model, the least-squares problem for the determination of the five astrometric parameters was:

$$\min \;\|\,\mathbf{A}_i\;\delta\mathbf{a}_i - \delta\tilde{\mathbf{v}}_i\,\|_2 \qquad [11.43]$$

which is to be solved once for each star. The software for the astrometric parameter determination included a number of tests and specialised algorithms for the weighting of the observations, the recognition of outliers, and the correction of grid-step errors. It also allowed a number of alternative models to be tested in addition to the standard one with only the five astrometric parameters $\lambda$, $\beta$, $\pi$, $\mu_{\lambda*}$, $\mu_\beta$, such as introducing an accelerated motion, or solving for the astrometric parameters of the centre of mass of double stars with known orbits.

For the double and multiple stars the abscissae were corrected for the duplicity effect as explained in Section 13.3, i.e. in such a way that the modified abscissae referred to the primary or to the photocentre of the binary, depending on the separation. The solution for the astrometric parameters of the primary or photocentre then proceeded in the same way as for the single stars.

### Weighting Scheme

One of the most important aspects of the least-squares solution for the five astrometric parameters was the scaling of the variances resulting from the great-circle reductions. Each observation equation was initially weighted by $w_{ji} = 1/\sigma^2_{v_{ji}}$, where, as before, $j$ stands for the circle and $i$ for the star, and $\sigma_{v_{ji}}$ was the standard deviation of the abscissa estimated by the great-circle reduction. In the FAST treatment several changes were brought to these standard deviations in order to scale the observation equations correctly.

For a given weighting scheme, the unit-weight variance for a particular star $i$ was computed as:

$$u_i^2 = \frac{1}{M_i - 5} \sum_{j \in \mathcal{J}_i} w_{ji} (\tilde{v}_{ji}^{\text{obs}} - v_{ji}^{\text{calc}})^2 \qquad [11.44]$$

where $\mathcal{J}_i$ is the set of reference great circles in which the star was included and $M_i$ is the number of such circles; $\tilde{v}_{ji}^{\text{obs}}$ is the observed abscissa, corrected as in Equation 11.42. The unit-weight variance should follow the distribution of the normalized chi-square variable $\chi^2_{M_i-5}/(M_i - 5)$ with unit mean. The sample distribution of $u_i^2$ was studied for various subsets of single and multiple stars as a function of magnitude and colours and led to a rather complex weighting system with $w_{ji} = 1/\sigma_{ji}^2$ depending on whether the star was single or double. For the stars processed as single, $\sigma_{ji}$ was computed as:

$$\sigma_{ji} = (0.86 + 0.0084\, Hp)(\sigma^2_{v_{ji}} + \sigma^2_m)^{1/2} \qquad [11.45]$$

where the additional standard deviation depending on the magnitude $Hp$ was given by:

$$\sigma_m = \begin{cases} 1.561\ (1 + 0.0978x + 0.0217x^2 + 0.0048x^3 + 0.0011x^4)\ \text{mas} & \text{if } Hp < 11.5 \\ 5\ \text{mas} & \text{otherwise} \end{cases} \qquad [11.46]$$

with:

$$x = 10^{(Hp-8)/5} - 1 \qquad [11.47]$$

This scheme was also used for weighting the equations of the primary reference stars in the sphere solution. For the double stars the corresponding expression was:

$$\sigma_{ji} = (0.86 + 0.028\, Hp)(\sigma^2_{v_{ji}} + \sigma^2_c)^{1/2} \qquad [11.48]$$

where $\sigma_c$ was the standard deviation of the correction applied to the abscissa in order to move the reference point to the primary (for separations $\varrho > 0.35$ arcsec) or to the photocentre (for $\varrho < 0.35$ arcsec) of the double star (see Chapter 13).

### Filtering of Outliers

There were essentially three modes for selecting or rejecting the observations:

1. the great-circle reduction provided several flags for every star observed in a circle to report on problems with the solution. The flagging was based on the statistical analysis of the residuals and most problems were connected to grid-step inconsistencies in the circle adjustment of the grid-abscissae. Out of nearly $3 \times 10^6$ abscissae, this led to the rejection of 22 000 observations, or about 0.7 per cent of the total;

2. from the study of the residuals of the abscissae it was possible to locate outliers with residuals larger than three times the standard deviation $\sigma_{ji}$. Then a new solution was computed until no more observations were rejected. For any star the fraction of rejected observations was kept below 30 per cent. The most general situation was no rejection at all (90.2 per cent of the stars) or only one rejection (7.6 per cent of the stars); only in 2.2 per cent of the cases were there two or more outliers. On the average there were just above three outliers per great-circle reduction, but this number was subject to considerable variation because the rejections were quite often concentrated on a few bad circles with problems of attitude convergence;
3. a manual mode with an *a priori* rejection of great circles based on a look-up table, mainly for the purpose of comparison or to study the influence of a particular configuration. The look-up table was specific for each star to be tested, and the software could be run only for a preselected set of stars.

### Correction of Grid-Step Errors

An algorithm to recognise and correct grid-step errors was devised by Bastian (1985). Its implementation worked smoothly, and was of constant use in the preliminary versions of the software. Its efficiency was however limited to circumstances when the proportion of great circles to be corrected was small and $|n_{ji}| \leq 2$ or 3. This was clearly unsatisfactory for many double stars and for the few hundred single stars with large errors in the Input Catalogue.

An alternative algorithm was therefore implemented. This searched for solutions at distances as large as 20 arcsec from the reference position. The software was a specialised version of the algorithm used to determine the relative astrometry of double stars in the FAST processing (Chapter 13). Indeed, the double star algorithm is, to an essential part, a robust grid-step error solver. In Equations 13.19–13.21 for the relative astrometry of double stars, the abscissa difference $\delta\tilde{v}_{ji}$ was substituted for the projected phase difference between the secondary and primary components. The solution for the 'double star parameters' $X = \varrho \sin\theta$ and $Y = \varrho \cos\theta$ then provided the desired update of the reference position.

Adding a parameter for the parallax, straightforward modifications led to a new method for the astrometric parameter determination, which were much less sophisticated than the nominal method, but very useful for producing a solution within a few milliarcsec of the true position, whatever the starting value. All the stars were therefore first solved with this alternative method, and the results then became the starting points for the actual astrometric parameter determination, in which there was no longer any grid-step problem.

### Practical Implementation

All the user-defined settings, combined with the possibility of running the program on a star by star basis, enhanced considerably the flexibility of the astrometric parameter software compared to the extreme rigidity of the sphere solution and proved to be decisive in the solution of all the non-trivial cases.

Two versions of a software originally developed at the Astronomisches Rechen-Institut in Heidelberg (Walter *et al.* 1985, Lenhardt *et al.* 1991) were implemented and run at two places. The evolution of the two versions was not fully parallel and the differences noticed in the results from time to time had to be carefully investigated. Eventually all the stars were processed on a single computer to produce the final FAST solution. Many intermediate cross-checks between CERGA and ARI helped make the final result very reliable. Also during this final step, frequent comparisons were made with the astrometric parameters of the primary reference stars computed during the sphere solution proper, which proved very useful for the understanding of the whole process.

## 11.7. Rank Deficiency and Convergence Properties

As mentioned in Section 11.1, the zero point corrections $c_j$ were determined in such a way that the corrected abscissae defined a globally consistent reference frame, but the observations themselves did not define any specific reference frame. This means that if a particular solution $\mathbf{a}$, $\mathbf{c}$, $\mathbf{\Gamma}$ to the least-squares problem of Equation 11.24 was found, then there existed an infinite number of (slightly) different solutions $\mathbf{a} + \delta\mathbf{a}$, $\mathbf{c} + \delta\mathbf{c}$, $\mathbf{\Gamma} + \delta\mathbf{\Gamma}$, for which the norm remained at the minimum. As a consequence the least-squares equations were expected to have a rank deficiency corresponding to the six degrees of freedom of the reference frame (Betti & Sansò 1983).

Contrary to this expectation it was found, already in the early simulations of the Hipparcos data reductions, that the equations for the sphere solution were in fact only weakly ill-conditioned (Lindegren & Söderhjelm 1985). This problem of the (absence of) rank deficiency was discussed at length in the Hipparcos literature (e.g. van Daalen, Bucciarelli & Lattanzi 1986). The conclusion has been that the non-singularity is due to the splitting of the overall problem into different steps, during which different parts of the unknowns of the problem were considered to be fixed. In this section the problem is re-analysed in the framework of the present formulation of the general problem, and the results of numerical experiments towards a more rigorous global solution of the astrometric parameters are described.

### Analysis of the Rank Deficiency

For simplicity the global parameters are excluded from the present discussion, as they are not expected to contribute in any significant way to the question of the rank deficiency. Furthermore, only one great-circle parameter was considered, i.e. $c_j$ or $\theta_{\mathrm{R}j}$. (Clearly the addition of more unknowns, such as the global parameters, cannot render the problem less ill-conditioned, and could therefore not be the source of the non-singularity of the actual equations.) The expected rank deficiency would consequently lead to the existence of non-zero vectors $\delta\mathbf{a}$ and $\delta\mathbf{c}$ satisfying:

$$\mathbf{A}\,\delta\mathbf{a} + \mathbf{C}\,\delta\mathbf{c} = \mathbf{0} \qquad [11.49]$$

For a particular observation this can be written:

$$\mathbf{d}'\delta\mathbf{a} - \delta\theta_{\mathrm{R}} = 0 \qquad [11.50]$$

where $\delta\mathbf{a}$ now refers to the one star in question.

This manifold of valid solutions to the least-squares problem corresponds to a set $S$ of reference frames differing from each other by a time-dependent orientation vector $\boldsymbol{\varepsilon}(t)$. Since the objective is to study the effects of small variations in the unknowns, it will be assumed that the orientation differences are small; thus, only first-order terms in the small quantities $\boldsymbol{\varepsilon}$, $\delta\mathbf{a}$, and $\delta\boldsymbol{\theta}$ are retained. Only a linear variation of $\boldsymbol{\varepsilon}$ with time can be absorbed by the proper motion components of the astrometric parameters; the time dependence must therefore be of the form:

$$\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}_0 + \boldsymbol{\omega} t \qquad [11.51]$$

The six degrees of freedom correspond to the components of $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\omega}$.

Let $[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,]$ be an arbitrary reference frame in the set $S$. Any other reference frame in $S$ can be written as $[\,\mathbf{x}+\delta\mathbf{x}\ \ \mathbf{y}+\delta\mathbf{y}\ \ \mathbf{z}+\delta\mathbf{z}\,]$, where $\delta\mathbf{x} = \boldsymbol{\varepsilon}\times\mathbf{x}$ etc. Since the direction to the star is independent of the reference frame, $\delta\mathbf{r} = \mathbf{0}$ and Equation 11.4 gives:

$$\mathbf{p}\,\delta\lambda* + \mathbf{q}\,\delta\beta = \mathbf{r}\times\boldsymbol{\varepsilon} \qquad [11.52]$$

Inserting this into Equation 11.8a and multiplying by $\sec r$ gives for the first term in Equation 11.50:

$$\begin{aligned}\mathbf{d}'\delta\mathbf{a} &= \mathbf{m}'[\mathbf{r}\times\boldsymbol{\varepsilon}]\ \sec r = (\mathbf{m}\times\mathbf{r})'\boldsymbol{\varepsilon}\ \sec r\\ &\simeq [(\mathbf{R}\times\mathbf{u})\times\mathbf{u}]'\boldsymbol{\varepsilon}\ \sec^2 r \qquad [11.53]\end{aligned}$$

where, in the last step, Equation 11.7 was used with $|\mathbf{R}\times\mathbf{u}| = \cos r$ and $\mathbf{r}\simeq\mathbf{u}$ to first order in the small angles. For the second term in Equation 11.50 it is noted, from Equation 11.14, that $\theta_{\rm R} = \tilde{\mathbf{P}}'\mathbf{Q}$; thus:

$$\begin{aligned}\delta\theta_{\rm R} &= \tilde{\mathbf{P}}'\delta\mathbf{Q} = \tilde{\mathbf{P}}'(\boldsymbol{\varepsilon}\times\mathbf{Q}) = (\mathbf{Q}\times\tilde{\mathbf{P}})'\boldsymbol{\varepsilon}\\ &\simeq -\mathbf{R}'\boldsymbol{\varepsilon} \qquad [11.54]\end{aligned}$$

Here, again, the small-angle approximation was invoked for the last step. It should be noted that $(\lambda_{\rm R},\ \beta_{\rm R})$ are interpreted as invariants, so that $\delta\mathcal{R} = \boldsymbol{\varepsilon}\times\mathcal{R}$; on the other hand, $\tilde{\mathcal{R}}$ is effectively fixed by the great-circle reductions and therefore unaffected by $\boldsymbol{\varepsilon}$.

Combining Equations 11.53 and 11.54 gives:

$$\begin{aligned}\mathbf{d}'\delta\mathbf{a} - \delta\theta_{\rm R} &= \left[(\mathbf{R}\times\mathbf{u})\times\mathbf{u} + \mathbf{R}\cos^2 r\right]'\boldsymbol{\varepsilon}\ \sec^2 r\\ &= \left[\mathbf{u}\mathbf{u}'\mathbf{R} - \mathbf{R}\sin^2 r\right]'\boldsymbol{\varepsilon}\ \sec^2 r\\ &= (\mathbf{u} - \mathbf{R}\mathbf{R}'\mathbf{u})'\boldsymbol{\varepsilon}\ \tan r\sec r\\ &= (\mathbf{m}\times\mathbf{R})'\boldsymbol{\varepsilon}\ \tan r \qquad [11.55]\end{aligned}$$

where the vector triple product $[(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}\mathbf{c}'\mathbf{a} - \mathbf{a}\mathbf{b}'\mathbf{c}]$ was applied twice, and $\mathbf{R}'\mathbf{u} = \sin r$ was also used.

It is seen that Equation 11.50 is not strictly satisfied by the variations $\delta\mathbf{a}$, $\delta\theta_{\rm R}$ produced by a small rotation of the reference frame. In Equation 11.55 the right-hand side is of the order of $\tan r$ times the terms on the left-hand side. The condition number of the observation equations, instead of being infinite, should therefore be of the order of $|\tan r|^{-1}\simeq 10^2$, and the condition number of the normal equations should be $\kappa\simeq 10^4$. This is in fair agreement with what was found in the actual solutions (Section 11.4).

As suggested by previous studies, the reason for the non-singularity can be traced back to the approximation made in connection with Equation 11.15a, namely that the terms containing $\theta_{\rm P}$ and $\theta_{\rm Q}$ were neglected. Since $\delta\theta_{\rm P} = -\mathbf{P}'\boldsymbol{\varepsilon}$ and $\delta\theta_{\rm Q} = -\mathbf{Q}'\boldsymbol{\varepsilon}$, the neglected terms amount to:

$$\begin{aligned}(\delta\theta_{\rm P}\cos v + \delta\theta_{\rm Q}\sin v)\tan r &= -(\mathbf{P}\cos v + \mathbf{Q}\sin v)'\boldsymbol{\varepsilon}\tan r\\ &= -(\mathbf{m}\times\mathbf{R})'\boldsymbol{\varepsilon}\tan r \qquad [11.56]\end{aligned}$$

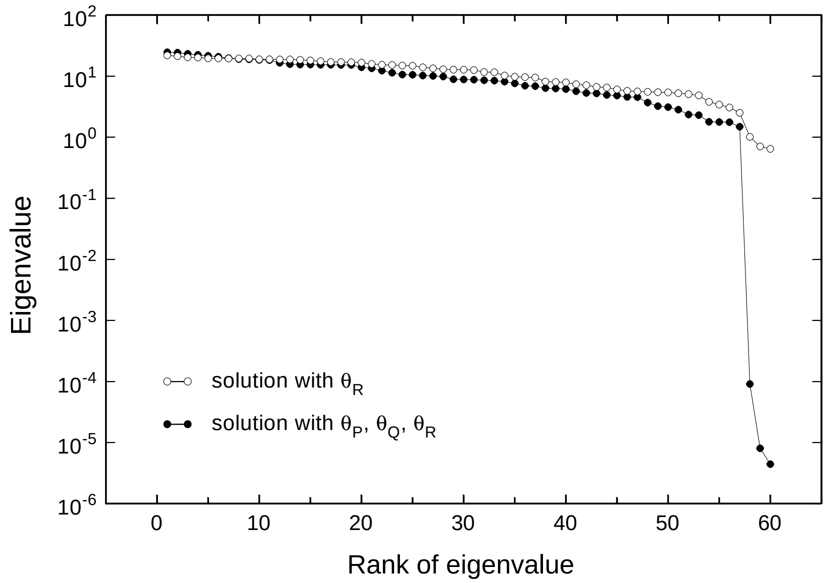


***Figure 11.3.*** *Eigenvalues for two small-scale simulations of the sphere solution, using 20 stars with 60 astrometric unknowns (positions and parallaxes). Open circles: only the abscissa zero point was estimated for each reference great circle. Filled circles: all three orientation parameters in* $\boldsymbol{\theta}$ *were estimated for each great circle. The expected rank deficiency of three shows up only in the latter case.*

exactly cancelling the previously found inequality. The inclusion of the two additional unknowns $\theta_{\rm P}$ and $\theta_{\rm Q}$ for each great-circle frame should therefore in principle provide the expected rank deficiency; in reality it should at least drastically increase the condition number of the design matrix.

## Numerical Experiments

One of the first numerical studies of the rank deficiency problem was performed by S. Söderhjelm in 1983. The observations of only 20 stars were simulated, assuming the nominal scanning law but with a 30° field of view. The positions and parallaxes were included as unknowns, together with one ($\theta_{\rm R}$) or three ($\boldsymbol{\theta}$) orientation parameters for each reference great circle. In this case the orientation parameters, rather than the astrometric parameters, were eliminated from the full normal equations, leading to reduced systems with $3N_p = 60$ unknowns. The eigenvalues of these systems are shown in Figure 11.3. The use of a single orientation parameter per great circle gave a rather well-conditioned system (open circles; condition number $\kappa \simeq 35$) while elimination of all three orientation parameters gave a very distinct jump from the 57th to the 58th ranked eigenvalue (filled circles; condition number $\kappa \simeq 5 \times 10^6$. This latter behaviour was exactly as expected for a well-posed least-squares problem with a rank deficiency of three, considering that single-precision arithmetics (four-byte reals) was used.

The sphere solutions performed by both reduction consortia used the formulation of Sections 11.3 and 11.4, including the approximation leading to the non-singularity of the least-squares problem. In a sense this was tantamount to injecting *a priori* positional information into the observation equations, forcing the poles of the actual great-circle frame to coincide with the nominal poles. As a consequence of this approach, the consistency of the final Hipparcos reference frame could in principle be spoiled by overconstraining (Lattanzi, Bucciarelli & Bernacca 1990). The external iteration

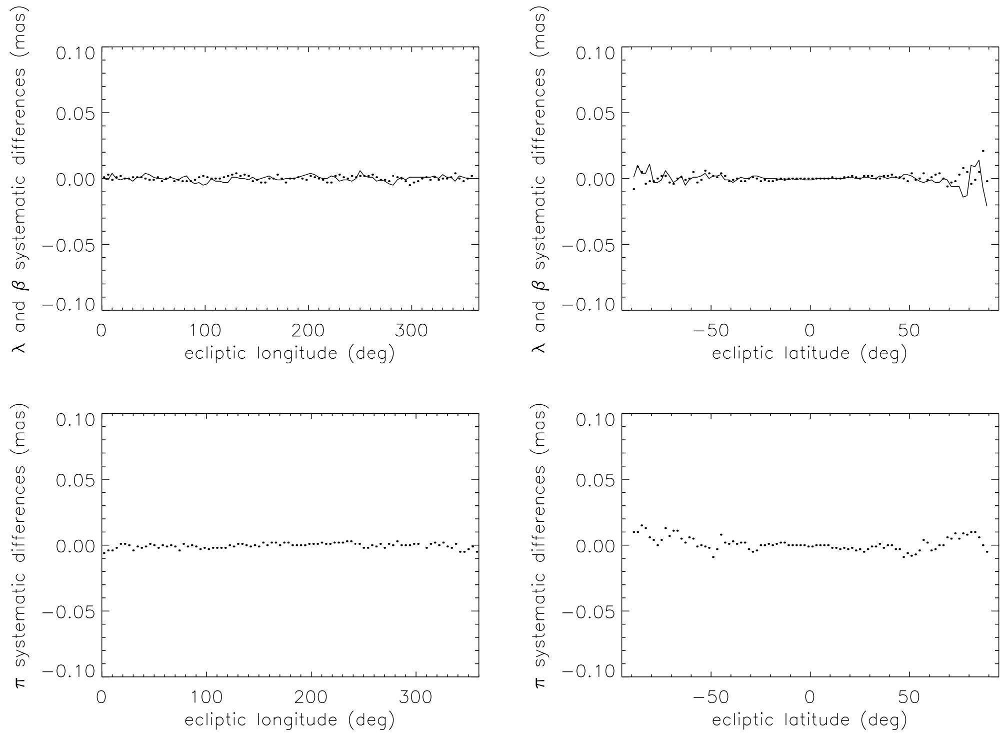


***Figure 11.4.*** *Residual systematic differences, estimated by the method of infinitely overlapping circles, between the modified solution (solving also for the longitudes of the reference great-circle poles) and the standard FAST sphere solution. The top panels show differences in ecliptic longitude (solid lines) and latitude (dotted), the bottom panels show the parallax differences.*

scheme adopted by the consortia (Section 16.2) was supposed to take care of this problem. However, it was not obvious that this procedure converged to a reference frame completely free of the distortion possibly introduced by the overconstraining; nor was it clear whether the relatively few iterations actually performed were sufficient for convergence.

A study of the convergence properties of the Hipparcos sphere solution, performed by B. Bucciarelli, M.G. Lattanzi and M. Frœschlé, compared the 37-month standard FAST solution before the last iteration with the corresponding results obtained by introducing the poles of the reference great circles as additional unknowns. Because of the low estimability of the adjustment to the latitude of the pole ($\Delta\beta_R$), the actual experiment was carried out with only one additional unknown per great circle, i.e. the adjustment to the longitude of the pole ($\Delta\lambda_R = \mathbf{z}'\boldsymbol{\theta}$). Its coefficient in the modified condition equation reads:

$$e_{\lambda_\mathrm{R}} = [\sin\nu\cos(\lambda_\mathrm{R}-\lambda) + \cos\nu\sin\beta_\mathrm{R}\sin(\lambda_\mathrm{R}-\lambda)]\cos\beta \qquad [11.57]$$

where $(\lambda, \beta)$ is the geometric position of the star.

Before comparing the modified solution to the FAST standard solution, a small rotation was applied to bring the former onto the system defined by the standard solution. The estimated rotation parameters were:

$$\begin{aligned} \varepsilon_x &= -0.3499 \pm 0.0002 \text{ mas} \\ \varepsilon_y &= -0.2445 \pm 0.0002 \text{ mas} \\ \varepsilon_z &= +0.0680 \pm 0.0002 \text{ mas} \end{aligned} \qquad [11.58]$$

The catalogue-wide rms of the positional differences were $\simeq 0.3$ mas and $\simeq 0.04$ mas in $\lambda$ before and after the rigid rotation, respectively; analogously, for $\beta$ the rms differences were $\simeq 0.2$ mas and $\simeq 0.03$ mas.

The method of infinitely overlapping circles (see Section 16.6) was utilised to evaluate residual systematic differences in the astrometric parameters. As both the modified and standard solutions were based on the same subset of 45 035 primary reference stars, the radius of the small circles was increased to $R = 3°$. This resulted in an average of 30 stars per circle. Figure 11.4 shows the computed systematic differences in position and parallax as functions of ecliptic longitude and latitude; a similar behaviour was observed for the proper motion differences. The systematic differences, at this resolution, are typically on the level of 0.01 to 0.02 mas. These results show that the two solutions are practically identical and support the conclusion that the external iterative scheme adopted by the consortia has been adequate to completely recover the errors in the *a priori* determined coordinates of the poles of the reference great circles.

Evidently this experiment cannot address possible distortions introduced earlier in the reduction procedure. To this end, a new reduction, which directly solves for the attitude parameters along with the astrometric parameters, would be required.

L. Lindegren, M. Frœschlé, F. Mignard